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Research Article

Bilevel optimization for the regulation of shared resources under uncertainty

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Abstract: We study normal-form noncooperative games corresponding to Common-Pool Resource (CPR) situations. A social planner aims to regulate the use of the resource and acts as a leader in a two-stage game. The followers are non-cooperative players who choose a Nash equilibrium representing optimal extraction levels. Due to uncertainty about the state of the resource, in this article we consider the possibility that production, which depends on the aggregation of strategies, is a set-valued mapping. We examine appropriate selections of this mapping and analyze the corresponding two-stage games where a social planner regulates the use of the resource and act as a leader. We prove the existence of a Stackelberg-Nash equilibrium and discuss its properties.

Keywors: Leader-follower model, Stackelberg-Nash equilibrium, Correspondence selections, Potential games

1. Introduction

In a non-cooperative game setting, the Common-Pool Resource (CPR) problem models a scenario in which individual players make extraction decisions independently without binding agreements, as is the case with shared groundwater and aquifers, fisheries, pastures, etc. This strategic context leads directly to over-exploitation, a phenomenon widely known as the Tragedy of the Commons [1]. A common-pool resource possesses two distinct economic characteristics: rivalry, since extraction by one user directly reduces the amount of the resource available to all others, and low excludability, since it is difficult or costly to prevent individuals from accessing and exploiting the resource (see [2-4]). Experimental evidence has documented strategic destruction of common-pool resources [2], while recent groundwater studies examine strategic extraction and groundwater exploitation [5, 6]. Recent experimental work further considers how ecological uncertainty and institutional arrangements affect the management of local commons [7-9].

The common-pool resource problem considers a fixed number N of actors who can access the resource ($N \in \mathbb{N}$, $N \geq 2$). For each actor i, $x_i \in [0, e]$ ($e \in \mathbb{R}$, $e > 0$) represents the amount invested by player i in the CPR, that is, the extraction effort, and we denote by

$$X = \sum_{i=1}^{N} x_i \tag{1}$$

the aggregate amount invested in the resource. The scalar $e > 0$ is an extraction-capacity bound.

The state of the resource is represented by the so-called

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production function H: $[0, Q] \to \mathbb{R}$, for a suitable value Q ($Q \gg N \cdot e$), which depends on the aggregate amount invested in the CPR, that is X. Typically, in applied examples, the function H is decreasing over its domain; however, our results hold in a general context. This function H captures congestion and competition in the consumption of the CPR, and thus the health of the resource or simply its state. The values of H are interpreted as the average productivity of the resource, such as the availability and ease of access to water in an aquifer.

Each agent derives a private benefit $U_i(x_i)$ from extracting a quantity x_i. In the non-cooperative CPR management scenario, each actor chooses an individual extraction level to maximize utility, which is proportional to the individual agent's effort and the efficiency generated by the collective, net of the purely individual cost:

$$x_i H\left(\sum_{j=1}^{N} x_j\right) - U_i\left(x_i\right). \tag{2}$$

Many strategic-form games in the literature exhibit this payoff structure, as the classical Cournot oligopoly, where H is given by the inverse demand function [10], or the traffic routing network games, where H represents the arc latency function, which depends on the total aggregate traffic [11].

Although most of the results in this paper hold for a convex, increasing function U_i, we adopt a private cost for user i proportional to the effort x_i, based on a positive real parameter ω. These games belong to the well-known class of aggregative games, in which the outcome depends on one's own actions and the aggregate of all players' actions. Many common games in industrial organization, public economics, and macroeconomics are aggregative games and have been studied extensively (see, for example, [12]).

In this paper we study a Stackelberg-Nash model for the regulation of a CPR, i.e. a two-level non-cooperative game in which a leader regulates extraction activities and the followers are users of a public good resource. The leader, who is the social planner, decides on a subsidy, intended as the cost of building infrastructure, to mitigate the consumption of the resource. By way of example, consider the situation of Managed Aquifer Recharge (MAR), in which the Water Basin Authority is the leader that invests in managed aquifer recharge infrastructure, such as stormwater infiltration basins or diversion channels.

Let $t \in [0, T]$ denote the value of the investment, for an appropriate real $T > 0$ which directly alters the physical state of the resource: if the authority chooses the public intervention t and $(x_1, ..., x_N)$ is the profile of withdrawal levels chosen by users, the state of the resource is given by

$$H\left(t + \sum_{i=1}^{N} x_i\right). \tag{3}$$

The hierarchical decision-making framework is modeled as a two-level optimization problem, in which a social planner (the regulatory authority or leader) maximizes social welfare and sustainability, under the constraint that users act as followers and maximize their individual utility by solving a parameterized Nash equilibrium problem: for every $t \in [0, T]$, the payoff of the i-th player is a function Π_i: $A \to \mathbb{R}$ defined by

$$\Pi_i\left(t, x_1, \ldots, x_N\right) = x_i H\left(t + \sum_{j=1}^{N} x_j\right) - \omega x_i, \tag{4}$$

where $A = [0, T] \times [0, e]^N$.

For each $t \in [0, T]$, we donote by Γ_t the N-player noncooperative game with payoff functions Π_i, $i = 1, \ldots, N$, and common strategy set $[0, e]$.

A Nash equilibrium profile of the game Γ_t is a vector $x^*(t) = (x_1^*(t), \ldots, x_N^*(t))$ such that, for each player $i \in \{1, \ldots, N\}$, $x_1^*(t)$ solves the following problem:

$$\sup_{x_i \in [0, e]} \Pi_i\left(t, x_i, x_{-i}^*(t)\right) \tag{5}$$

where $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$. The set of Nash equilibria of the parametrized game Γ_t is denoted by $NE(t)$ and the mapping $t \in [0, T] \rightrightarrows NE(t)$ is called the best reply correspondence.

If we include a social planner, we deal with a (N + 1)-players game in which the leader chooses $t \in [0, T]$ and the N follower users choose their own extraction level $x_i \in [0, e]$. The social planner does not limit resource extraction, but supports a level of public investment to build or maintain infrastructure aimed at subsidizing the extraction effort, while maximizing the payoff $W : A \to \mathbb{R}$ defined as the aggregate social welfare function minus a cost r: $[0, T] \times [0, Ne] \to \mathbb{R}$ as follows:

$$W\left(t, x_1, \ldots, x_N\right) = \sum_{i=1}^{N} \Pi_i\left(t, x_1, \ldots, x_N\right) - r(t, X). \tag{6}$$

Definition 1. A Stackelberg-Nash equilibrium is a (N + 1)-tuple

$$\left(t^*, x_1^*\left(t^*\right), \ldots, x_N^*\left(t^*\right)\right) \tag{7}$$

such that $t^* \in [0, T]$ solves the upper level problem:

$$\sup_{t \in [0, T]} W\left(t, x_1^*(t), \ldots, x_N^*(t)\right) \tag{8}$$

where, for each $t \in [0, T]$, $x^*(t) = (x_1^*(t), \ldots, x_N^*(t))$ is the unique solution of the lower level problem

$$\sup_{x_i \in [0, e]} \Pi_i \left(t, x_i, x_{-i}^*(t)\right), \ \forall i = 1, 2, \ldots, N. \tag{9}$$

Note that we assume that problem (9) has a unique solution. It is possible to avoid this assumption, using suitable definitions of the Stackelberg problem assuming that the leader has an optimistic or a pessimistic behavior. This case is not considered in this article.

In Definition 1, (8) is called the upper level optimization problem and (9) denotes the lower-level optimization problem in a two-level formulation. This model is known in the literature as a Stackelberg-Nash equilibrium with one leader and many followers [10, 13, 14]. The regulatory authority anticipates how lower-level users will react to its decision, leading to a perfect subgame equilibrium. Two-level optimization models are widely used to develop concrete environmental policies, such as in the management of CPRs, but also in many other contexts [10, 15, 16]. Recent work also develops decomposition methods for single-leader multi-follower games and equilibrium models under decision-dependent uncertainty [13, 14].

The aggregate return on investment in the CPR plays a key role in the model. As is often the case in real-world situations, there is uncertainty regarding revenues due to externalities. The classic example is the global emissions game, where, as is well known, there is considerable scientific uncertainty regarding the impact of gas accumulation on Earth's temperature.

In this work, we will present a CPR model in which an agent perceives the impact of contributors on the CPR under conditions of uncertainty. In [17] uncertainty consists of two possible states of the world: an optimistic one and a pessimistic one, and agents differ in their beliefs regarding the probability of one state relative to the other. An agent assigns a probability to the true optimistic state and optimizes the state-dependent expected payoff. The analysis is carried out by studying the corresponding stochastic game.

Our setting differs in the source and treatment of uncertainty. We do not assume that a probability distribution is known, nor do we impose a worst-case objective. Instead, we represent the physical state itself as a multifunction and use economically motivated selections. In this article, we establish existence results of the bilevel problem under set-valued uncertainty, namely when the state of resources $H: [0, Q] \rightrightarrows \mathbb{R}$ is a set-valued function: if player i invests the amount x_i, the aggregate investment of the other players is $\sum_{j \neq i} x_j$ and the social planner chooses the control variable t, the production is given by any value in the set $H(t + x_i + \sum_{j \neq i} x_j)$ and the profit Π_i is set-valued as well. The game is framed in the context of set-valued optimization [18].

This formulation is particularly natural when measurements, environmental heterogeneity, or model uncertainty prevent the authority from assigning a unique state to a given aggregate extraction level.

The important methodological point is that there is no universally mandatory way to convert such uncertainty into a decision model. Fuzzy formulations require a possibility/possibilistic structure; interval approaches retain interval-valued outcomes and therefore require a decision rule for comparing interval solutions; stochastic formulations require a probability distribution or probabilistic model. Robust formulations instead specify a worst-case criterion over an uncertainty set. These paradigms answer different questions [15, 16, 19]. Fuzzy formulations of equilibrium problems are studied in both noncooperative and leader-follower settings [20, 21], while recent work also considers set-valued equilibrium problems [22]. We deliberately choose another route: the set-valued state is retained at the modeling stage, and a selection is then chosen according to the economic purpose of the analysis.

We investigate two selections. First, the maximum selection

$$h(z) = \max H(z) \tag{10}$$

is natural when the relevant criterion is profit maximization and the decision maker selects the most profitable admissible state. Second, when a simple and computationally transparent representative state is desired, we use a linear selection obtained from the endpoint information of an interval-valued state. The latter converts the follower game into an exact potential game with a unique Nash equilibrium under mild assumptions.

This is not intended to claim that the linear selection is the only reasonable representation; rather, it is a deliberate modeling choice that makes the equilibrium analytically and computationally transparent.

Table 1 summarizes the distinction.

In section 2, as profit maximizers, we consider the bilevel problem and provide existence results for the maximum selection approach and the linear selection one. In section 3, we develop a groundwater-management application together with sensitivity and computational experiments. Section 4 summarizes these studies and suggests some urther directions for research.

Table 1. Position of the present approach relative to common uncertainty paradigms

Approach	Information/structure required	Typical decision interpretation
Fuzzy	Possibility or membership information	Decisions based on degrees of possibility/membership
Interval	Bounds for uncertain quantities	Interval-valued outcomes; an additional comparison/ selection rule is generally needed
Stochastic	Probability distribution or probabilistic model	Expected, chance-constrained, risk-sensitive, or distribution-based decisions
Robust	Uncertainty set and worst-case criterion	Protection against adverse realizations
Present approach	Set-valued state mapping H	A deliberate selection: max-selection for profit maximization or linear selection for analytical tractability and unique Nash equilibrium

2. Selection under set-valued uncertainty

2.1 Maximum selection

We recall some notions concerning the set-valued mappings. In the following we will consider maps with non empty values, among metric spaces Z and Y. A set-valued map $H: Z \rightrightarrows Y$ is called closed at $z \in Z$, if for any $z_n \to z$ and $y_n \to y$ verifying $y_n \in H(z_n)$, $\forall n \in \mathbb{N}$, then $y \in H(z)$ [23]. The map H is called subcontinuous at $z \in Z$ if, given $z_n \to z$, any $(y_n)_n$ such that $y_n \in H(z_n)$, $\forall n \in \mathbb{N}$, has a subsequence converging to a point $y \in Y$. The map H is closed (resp. subcontinuous) on Z if it is closed (resp. subcontinuous) at any $z \in Z$. Let us remark that if Y is compact, then H is subcontinuous on Z, while, for general Y, a compact-valued map H (i.e. $H(z)$ compact for any z) is not necessarily subcontinuous. Moreover, if H is subcontinuous and closed at z, the set $H(z)$ is compact. Indeed, let $y_n \in H(z)$, for $n \in \mathbb{N}$. As H is subcontinuous, (for a not relabeled subsequence) $y_n \to y$. On the other hand, since H is closed, we conclude $y \in H(z)$.

Given a map H: $[0, Q] \rightrightarrows \mathbb{R}$, a selection of H is a function h: $z \in [0, Q] \to h(z) \in H(z)$. We are interested in the max-selection, keeping in mind that the players seek to maximize profit.

Definition 2. Let H: $[0, Q] \rightrightarrows \mathbb{R}$ be compact valued. The max-selection of the map H is defined as the function h: $z \in [0, Q] \to h(z) \in \mathbb{R}$ where

$$h(z) = \max H(z). \tag{11}$$

Lemma 1. Assume H: $[0, Q] \rightrightarrows \mathbb{R}$ be closed and subcontinuous. Then, H is compact valued and the function h defined in (11) is upper semicontinuous.

Proof. As noted, $H(z)$ is compact for any $z \in Z$. We show that h is upper semicontinuous. Let $z_n \to z$. As H is subcontinuous, we may assume that $h(z_n)$ is converging, say $y = \lim h(z_n)$. Moreover, since H is closed, $y \in H(z)$. Therefore, $y \le h(z) = \max H(z)$.

This choice of Definition 2 corresponds to the optimistic view in [17]. Using the max-selection, we consider a parametric non-cooperative N-player game $\overline{\Gamma}_t$ as the lower level problem (9) in Definition 1 with payoffs

$$\overline{\pi}_i : (t, x_1, \ldots, x_N) \in A \to x_i h\left(t + \sum_{j=1}^{N} x_j\right) - \omega x_i. \tag{12}$$

The correspondence $t \in [0, T] \rightrightarrows NE(t)$ denotes the correspondence of the best responses with payoffs $\overline{\pi}_i$ for any $i \in \{1, ..., N\}$ as in (12).

Proposition 2 (The lower level problem). Let us suppose that the set-valued map H is closed and subcontinuous on $[0, Q]$. Then:

i) The function $(t, x) \in A \to h\left(t + \sum_{i=1}^{N} x_i\right)$ is upper semicontinuous on $A = [0, T] \times [0, e]^N$;

ii) The set of the Nash equilibria $NE(t)$ is not empty for any $t \in [0, T]$;

iii) The set-valued map $t \in [0, T] \rightrightarrows NE(t)$ is closed.

Proof. i) It is a direct consequence of Lemma 1.

ii) The parametric game $\overline{\Gamma}_t$ has a potential structure: it is, in fact, an ordinal potential game with potential function

$$P(t, x_1, \ldots, x_N) = x_1 x_2 \cdots x_N \left(h\left(t + \sum_{i=1}^{N} x_i\right) - \omega\right). \tag{13}$$

Indeed, for any $x_{-i} \in [0, e]^{N-1}$ and for any $x_i', x_i'' \in [0, e]$ we have

$$P(t, x_i', x_{-i}) - P(t, x_i'', x_{-i})$$

$$= (x_1 \cdots x_{i-1} x_{i+1} \cdots x_N)\left[\overline{\pi}_i(t, x_i', x_{-i}) - \overline{\pi}_i(t, x_i'', x_{-i})\right]. \tag{14}$$

Since the potential function P is upper semi-continuous on A, it has at least a maximizer, which is a Nash equilibrium of $\overline{\Gamma}_t$. [Let us recall that a game $\left\langle N; (X_i)_{i=1}^N; (\Pi_i)_{i=1}^N \right\rangle$

is an ordinal potential game if there exists a function $P:\prod_{i=1}^{N} X_i \to \mathbb{R}$ called potential function, such that for each player $i \in \{1, \ldots, N\}$, each strategy profile $x_{-i} \in \prod_{j \neq i} X_j$ of i's opponents, and each pair $x_i, y_i \in X_i$ of strategies of player i

$$\Pi_i(x_i, x_{-i}) - \Pi_i(y_i, x_{-i}) > 0$$

$$\Leftrightarrow P(x_i, x_{-i}) - P(y_i, x_{-i}) > 0 \tag{15}$$

and exact potential game if

$$\Pi_i(x_i, x_{-i}) - \Pi_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}). \tag{16}$$

In both cases, the set of the potential maximizers is contained in the set of the Nash equilibrium profiles of the game [24].]

iii) We prove that the map $t \in [0, T] \rightrightarrows NE(t)$ is closed at arbitrary $\hat{t} \in [0, T]$. Let $t_n \to \hat{t}$ and $x_n = (x_{1n}, \ldots, x_{Nn}) \in NE(t_n)$ for $n \in \mathbb{N}$, verify $x_n \to \hat{x} = (\hat{x}_1, \ldots, \hat{x}_N)$; we must show that $\hat{x} \in NE(\hat{t})$.

By upper semicontinuity we have $\limsup_n \overline{\pi}_i(t_n, x_n) \leq \overline{\pi}_i(\hat{t}, \hat{x})$; choose $\tilde{x}_i \neq \hat{x}_i$ and define the sequence

$$\tilde{x}_{in} = \hat{t} + \tilde{x}_i + \sum_{j \neq i} \hat{x}_j - \left(t_n + \sum_{j \neq i} x_{jn} \right) \tag{17}$$

converging to $\tilde{x}_i$ and such that $t_n + \tilde{x}_{in} + \sum_{j \neq i} x_{jn} = \hat{t} + \tilde{x}_i + \sum_{j \neq i} \hat{x}_j$. Then

$$\overline{\pi}_i\left(t_n, \tilde{x}_{in}, x_{-in}\right) = \tilde{x}_{in} h\left(t_n + \tilde{x}_{in} + \sum_{j \neq i} x_{jn} \right) - \omega \tilde{x}_{in}$$

$$= \tilde{x}_{in} h\left(\hat{t} + \tilde{x}_i + \sum_{j \neq i} \hat{x}_j \right) - \omega \tilde{x}_{in}. \tag{18}$$

Since $x_n \in NE(t_n)$ as any $n \in \mathbb{N}$, we have that $\overline{\pi}_i\left(t_n, \tilde{x}_{in}, x_{-in}\right) \leq \overline{\pi}_i\left(t_n, x_n\right)$: considering the limit for $n \to +\infty$ and by virtue of the upper semicontinuity of $\overline{\pi}_i$, we obtain $\overline{\pi}_i(\hat{t}, \tilde{x}_i, \hat{x}_{-i}) \leq \overline{\pi}_i(\hat{t}, \hat{x}_1, \ldots, \hat{x}_N)$, i.e. $\hat{x} \in NE(\hat{t})$. □

Taking into account formula (6) and the max-selection (11), the social planner optimizes the payoff

$$\overline{w}(t, x_1, \ldots, x_N) = \sum_{i=1}^{N} \left[x_i h\left(t + \sum_{j=1}^{N} x_j\right) - \omega x_i \right] - r(t, X) \tag{19}$$

by choosing the optimal value of $t \in [0, T]$ as indirect regulatory lever, subject to the constraint that the players involved in the consumption of the CPR invest $x(t) \in NE(t)$.

The function r in (19) represents the cost incurred by the social planner to build or maintain the regulatory infrastructure, as a function of the regulation parameter t and on the aggregate amount invested in the resource.

Proposition 3 (The upper level problem). Let us suppose that the set-valued map H is closed and subcontinuous on $[0, Q]$ and that r is a lower semicontinuous function on $[0, T] \times [0, Ne]$. If the set of the Nash equilibria is a singleton for any $t \in [0, T]$, $NE(t) = \{\hat{x}(t)\}$, then the problem

$$\sup_{t \in [0, T]} \overline{w}(t, \hat{x}(t)) \tag{20}$$

has a solution.

Proof. From Proposition 2 the function $t \to \hat{x}(t)$ is continuous on $[0, T]$, then $t \to \hat{X}(t) = \sum_{i=1}^{N} \hat{x}_i(t)$ is continuous as well.

Max-selection in example 1 is shown as Figure 1.

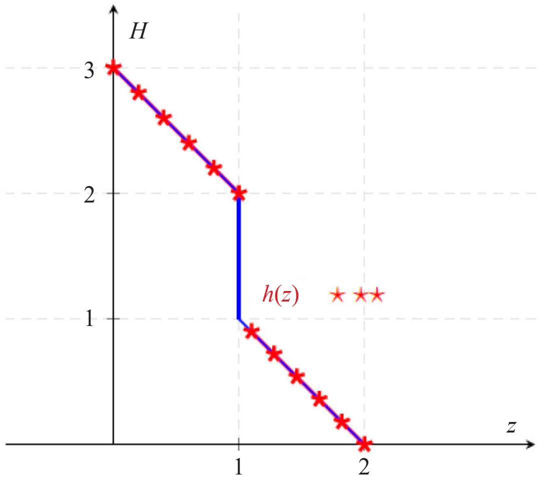

Figure 1. Max-selection in example 1

Example 1. Consider the case with two followers, where: $e = \frac{1}{2}$, $T = \frac{1}{2}$, $\omega = 1$ and the set valued status is given by

$$H(z) = \begin{cases} 3 - z & \text{if } 0 \leq z < 1 \\ [2, 3] & \text{if } z = 1 \\ 2 - z & \text{if } 1 < z \leq 2. \end{cases} \tag{21}$$

The max-selection of the map H is given by

$$h(z)=\begin{cases}3-z & \text{if } 0\le z\le 1\\ 2-z & \text{if } 1<z\le 2.\end{cases} \quad (22)$$

For any $t\in\left[0,\ \frac{1}{2}\right]$, the Nash equilibria of the followers are

$$NE(t)=\left\{(x_1,\ 1-x_2-t),\ x_1\in\left[\frac{1}{2}-t,\ \frac{1}{2}\right]\right\} \quad (23)$$

and the aggregate of the equilibrium strategies is

$$X^*(t)=1-t. \quad (24)$$

We assume that the social planner chooses a cost $r(t, X) = pX^2 - qt^2$, with $p, q > 0$ real parameters.

For $p=\frac{9}{5}$ and $q = 1$, the leader maximizes

$$W\left(t,\ x_1^*(t),\ x_2^*(t)\right)$$

$$=2X^*(t)-\frac{9}{5}\left(X^*(t)\right)^2+t^2$$

$$=\frac{1}{5}\left[-4t^2+3t+6\right]. \quad (25)$$

The best regulation parameter is $t^*=\frac{3}{8}$ and the followers will invest in the resource $\frac{5}{16}$ each.

For $p=q=\frac{1}{4}$, the leader maximizes

$$W\left(t,\ x_1^*(t),\ x_2^*(t)\right)=2X^*(t)-\frac{1}{4}\left(X^*(t)\right)^2+\frac{1}{4}t^2$$

$$=-\frac{5}{2}t+\frac{11}{4}. \quad (26)$$

The best regulation parameter in this case is $t^* = 0$ and the followers will invest in the resource $\frac{1}{2}$ each, i.e. to the maximum of their possibilities.

Remark 1. Note that, assuming a cost $r(t, X) = pX^2 - qt^2$, the leader derives a direct benefit from maintaining a high value of t, but must balance this benefit against the economic collapse of the users. The term pX^2 represents a strongly convex maintenance cost (congestion or wear-and-tear cost). The term qt^2 (savings in terms of efficiency or regulatory burden) represents a reduction in total cost: as the regulatory effort t increases, the infrastructure generates a form of "economies of scale" that reduces the overall financial burden on the leader; for example, the digitization of controls pays for itself.

If the social planner uses a cost $r(t,\ X)=\frac{1}{4}X^2-\frac{1}{4}t^2$, the leader's optimal choice *is* $t^* = 0$*:* when the regulatory authority sets $t = 0$, there is no longer any institutional pressure affecting the resource. Users, acting non-cooperatively and choosing a Nash equilibrium, maximize their private profit and, in the absence of conservation incentives ($t^* = 0$), adopt predatory individual behavior rationality takes precedence. This situation is known in the literature as the "tragedy of the commons": the social planner concludes that letting the market run its course (laissez-faire) generates a greater aggregate economic surplus for society than any attempt at rationing and therefore decides not to intervene, tax, or restrict the resource.

2.2 Linear selection

As a measure of the resource's state, the aggregate does not calculate an exact average, but estimates a confidence interval. In practical applications, it is natural to estimate that production falls within a certain range. We assume the state H to be interval-valued

$$H: x\in[0,\ Q]\rightrightarrows[\alpha(x),\ \beta(x)]\subset\mathbb{R}, \quad (27)$$

with α, β defined and decreasing on $[0, Q]$ such that $\alpha(x) \le \beta(x)$, $\forall x \in [0, Q]$, α convex, β concave on its domain.

These assumptions on α, β show that most of the uncertainty regarding the state of the resource does not lie in the extreme cases, that is, when the resource is nearly empty or almost full. Based on these assumptions, we can consider the following linear selection of the mapping H, namely the function $l: z\in[0,\ Q]\to l(z)\in\mathbb{R}$ where

$$l(z)=\left[\frac{\left(\alpha(Q)-\alpha(0)\right)+\left(\beta(Q)-\beta(0)\right)}{2Q}\right]z+\frac{\alpha(0)+\beta(0)}{2} \quad (28)$$

Let $a=\frac{\alpha(0)+\beta(0)}{2}$ and $b=-\left[\frac{\left(\alpha(Q)-\alpha(0)\right)+\left(\beta(Q)-\beta(0)\right)}{2Q}\right]$ $(a,\ b > 0)$. The assumptions regarding the lower bound α and upper bound β imply that l is a decreasing linear function and satisfies $\alpha(x) \le l(x) \le \beta(x)$ for any $x \in [0, Q]$. [It is sufficient to remark that $s_\beta(x) \le l(x) \le s_\alpha(x)$ for any $x \in [0, Q]$ where $s_\beta(x)$ is the line connecting $(0, \beta(0))$ and $(Q, \beta(Q))$, $s_\alpha(x)$ is the line connecting $(0, \alpha(0))$ and $(Q, \alpha(Q))$.]

Using the linear selection l, we consider a non-cooperative parametric N-player game $\tilde{\Gamma}_t$ as a lower-level (9) problem in Definition 1 with payoffs

$$\tilde{\pi}_i : (t, x_1, \ldots, x_N) \in A \to x_i l\left(t + \sum_{i=1}^{N} x_i\right) - \omega x_i$$

$$= x_i\left(a - bt - b\sum_{i=1}^{N} x_i\right) - \omega x_i. \quad (29)$$

For every $t \in [0, T]$, the mapping $t \in [0, T] \rightrightarrows NE(t)$ indicates the set of the Nash equilibria of the N-player noncooperative game $\tilde{\Gamma}_t$ where each player has profit $\tilde{\pi}_i$ as in (29) and strategy set $[0, e]$. Analogously to Proposition 2, we prove the following result.

Proposition 4 (The lower level problem). Assume that $l(0) = a > \omega$. For any $t \in [0, T]$, the game $\tilde{\Gamma}_t$ admits a unique Nash equilibrium and the best reply $t \in [0, T] \to NE(t) = \{\tilde{x}(t)\}$ is a continuous function.

Proof. The parametric game $\tilde{\Gamma}_t$ has a potential structure [24]: it turns out to be an exact potential game with potential function

$$P(t, x_1, \ldots, x_N)$$

$$= (a - bt - \omega)\sum_{i=1}^{N} x_i - b\sum_{i=1}^{N} x_i^2 - b\sum_{1 \le i < j \le N} x_i x_j \quad (30)$$

that is continuous on A. Then P has at least a maximizer that is Nash equilibrium of $\tilde{\Gamma}_t$. Under the assumptions, the unique maximizer of P that is also the unique Nash equilibrium

$$x_i^*(t) = \begin{cases} \dfrac{a - \omega - bt}{b(N+1)} & \text{if } 0 \le t \le \dfrac{a-\omega}{b} \\ 0 & \text{if } t > \dfrac{a-\omega}{b} \end{cases} \quad (31)$$

for any $i = 1, \ldots, N$ if $\dfrac{a-\omega}{b} < T$, otherwise $x_i^*(t) = \dfrac{a-\omega-bt}{b(N+1)}$, $\forall t \in [0, T]$. In any case the best reply $t \in [0, T] \to NE(t) = \{\tilde{x}(t)\}$ is a continuous function. □

The social planner optimizes the payoff

$$\tilde{w}(t, x_1, \ldots, x_N) = \sum_{i=1}^{N}\left[x_i l\left(t + \sum_{i=1}^{N} x_i\right) - \omega x_i\right] - r(t, X) \quad (32)$$

The bilevel problem admits a solution.

Proposition 5 (The upper level problem). Assume that $l(0) = a > \omega$ and that r is a lower semicontinuous function on $[0, T] \times [0, Ne]$. Then the problem

$$\sup_{t \in [0, T]} \tilde{w}(t, \tilde{x}(t)) \quad (33)$$

admits a solution.

Remark 2. The linear selection is not introduced as a claim that uncertainty has disappeared in the physical system. It is a decision rule that maps the uncertain state into a single representative state. Its specific advantage is that the resulting lower-level game has an explicit unique equilibrium and can be solved through potential maximization.

3. Aquifer management model

In this section, we present a case study with broad applicability: groundwater management, a problem that frequently arises in arid agricultural regions, where hundreds of private farmers N (followers) share a single underground aquifer. This situation can be modeled using the two-level model (4), in which a regional water resources management agency (social planner) uses a policy tool that alters the health status of the resource stock. In water economics, instead of rationing water based on volume (which is difficult to measure and control), the regulatory authority artificially inflates the energy cost per unit of extraction to simulate the presence of a deeper aquifer, by implementing $t \in [0, T]$, a real parameter that acts as a preventive energy surcharge. Every farmer seeks to maximize the private profit derived from his crop yield, net of their water pumping costs.

The strategies $x = (x_1, \ldots, x_N) \in [0, e]^N$ represent the level of private extraction or the intensity of resource exploitation by the users. The constraint ($x_i \in [0, e]$) indicates the existence of a physical or biological limit on maximum individual extraction capacity. The parameter t represents an environmental state variable as a multiplier of congestion or resource degradation factor controlled by the social planner and appears in the status function $H(t + X) = a - b(t + X),\ a, b > 0\left(X = \sum_{i=1}^{N} x_i\right)$. Assuming that the individual cost is proportional to the extracted resource ωx_i with ω real positive number, the farmer's payoff is

$$\Pi_i(t, x) = x_i\left[a - \omega - bt - bX\right]. \quad (34)$$

The equilibrium extraction is given by (31), and the aggregate extraction is

$$X^*(t) = \begin{cases} \dfrac{N(a - \omega - bt)}{b(N+1)}, & 0 \le t \le \dfrac{a-\omega}{b}, \\ 0, & t > \dfrac{a-\omega}{b}. \end{cases} \quad (35)$$

The planner solves

$$\max_{t\in[0,\,T]}\left\{X^*(t)[a-\omega-bt-bX^*(t)]-r(t,\,X^*(t))\right\}. \quad (35)$$

The model illustrates the distinction between the physical uncertainty and the regulatory decision. The authority does not need to observe an exact resource state before defining the selection; instead, the available state information is summarized by H, and the selected representation determines the equilibrium response used by the upper-level problem.

3.1 Numerical illustration and sensitivity

We present the illustrative two-farmer configuration

$$e=\frac{1}{2},\,T=1,\,\omega=\frac{1}{4}, \quad (37)$$

and consider

$$H(z)=\left[\alpha(z),\ \beta(z)\right], \quad (38)$$

with

$$\alpha(z)=\frac{(z-2.5)^2}{10},\,\beta(z)=\frac{15}{8}-\frac{3}{10}z^2,\ \ 0\le z\le 2.5. \quad (39)$$

The linear selection is

$$l(z)=\frac{5}{4}-\frac{z}{2}, \quad (40)$$

so $a = 5/4$ and $b = 1/2$. Formula (31) gives

$$x_1^*(t)=x_2^*(t)=\frac{2-t}{3},\,X^*(t)=\frac{2(2-t)}{3}. \quad (41)$$

For the illustrative regulatory cost

$$r(t,\,X)=X^2-X-\frac{t}{9}-\frac{13}{72}, \quad (42)$$

the induced leader objective is

$$W(t)=-\frac{2}{9}t^2+\frac{1}{3}t+\frac{45}{72}, \quad (43)$$

whose maximizer is

$$t^*=\frac{3}{4},\,x_1^*(t^*)=x_2^*(t^*)=\frac{5}{12}. \quad (44)$$

Status set-valued map H and the linear selection l in subsection 3.1 is shown as Figure 2.

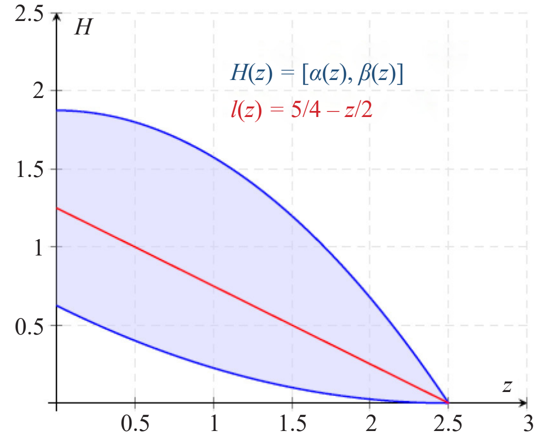

Figure 2. Status set-valued map H and the linear selection l in subsection 3.1

To address sensitivity, the relevant comparative-static quantities are immediate from (31). Before the extraction level reaches zero,

$$\frac{\partial x_i^*}{\partial t}=-\frac{1}{N+1},\ \frac{\partial X^*}{\partial t}=-\frac{N}{N+1}. \quad (45)$$

Moreover,

$$\frac{\partial x_i^*}{\partial a}=\frac{1}{b(N+1)},\ \frac{\partial x_i^*}{\partial \omega}=-\frac{1}{b(N+1)},\ \frac{\partial x_i^*}{\partial b}=-\frac{a-\omega}{b^2(N+1)} \quad (46)$$

for fixed t in the interior region. Hence stronger regulation reduces aggregate extraction, a higher baseline productivity increases extraction, and higher private extraction cost reduces extraction. These formulas also provide a transparent benchmark for larger numerical experiments.

In the same model, we can vary the number of followers over

$$N\in\{2,\,5,\,10,\,25,\,50,\,100\}. \quad (47)$$

For every N, the unique follower equilibrium is computed from

$$x_i^*(t)=\frac{a-\omega-bt}{b(N+1)}=\frac{2-t}{N+1},\ 0\le t\le 1, \quad (48)$$

and hence

$$X^*(t)=\frac{N(2-t)}{N+1}. \quad (49)$$

The numerical results are reported in Table 2.

Table 2. The follower equilibrium under the linear selection

N	$x_i^*(0)$	$X^*(0)$	$x_i^*(0.5)$	$X^*(0.5)$	$\max_t$ gross welfare
2	0.6667	1.3333	0.5000	1.0000	1.0000
5	0.3333	1.6667	0.2500	1.2500	1.3889
10	0.1818	1.8182	0.1364	1.5000	1.4876
25	0.0769	1.9231	0.0577	1.4423	1.5385
50	0.0392	1.9608	0.0294	1.4706	1.5483
100	0.0198	1.9802	0.0149	1.4851	1.4949

Individual extraction decreases as the number of followers increases, while aggregate extraction approaches the limiting value implied by the continuous-population benchmark (as N approaches $+\infty$, $X^*(t)$ approaches 2).

4. Concluding remarks

We have studied two-level models corresponding to a "one leader–many followers" problem, known as the Stackelberg-Nash equilibrium problem. Our analysis focused on special games, a particular class of aggregative games reducible to a Common-Pool Resource (CPR) problem.

The leader is the central authority that aims to regulate resource extraction. The followers are the resource users who compete in a non-cooperative game. As in the common-pool resource problem, their payoffs depend on the state of the resource, which is inherently subject to measurement and thus involves uncertainty. The leader's strategy incorporates the state function and maximizes the social welfare of the agents, net of the cost of constructing and/or maintaining the infrastructure that facilitates resource management.

Using the selection technique for the multifunction that represents the state, we present existence results for the two-level problem. The selections used in this article lead to a lower-level equilibrium problem that exhibits the following potential property: the calculation of the Nash equilibrium is based on a simple optimization problem, thereby reducing computational costs.

There are many other possible research directions: the use of the selection approach for a two-level problem with set-valued outcomes could be studied for a broader class of games. Another possibility is the study of a two-level problem with many leaders and many followers, which is also very interesting for applications. All these points will be the subject of future research.

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Author contributions

Luigi Greco: Conceptualization, Methodology, Formal analysis, Writing, review and editing. Lina Mallozzi: Conceptualization, Methodology, Formal analysis, Writing, review and editing. Both authors approved the final manuscript.

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Data availability

No datasets were generated or analyzed in this study. The numerical examples are fully specified in the manuscript.

Code availability

No external software package or proprietary code was required for the analytical results. The numerical calculations reported in the manuscript can be reproduced directly from the displayed formulas.

Conflicts of interest

The authors declare that they have no conflict of interest.

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