

Binary extended theta operation of binary soft sets

Aslıhan Sezgin, Orhan Karamustafaoğlu*

Department of Mathematics and Science Education, Faculty of Education, Amasya University, P.O. Box 05100, Amasya, Turkey

* Corresponding to: Orhan Karamustafaoğlu, Email: orhan.karamustafaoğlu@amasya.edu.tr

Abstract: Binary soft set theory, first introduced by Açıkğöz and Taş in 2016, has become widely accepted as a technique for addressing and modeling uncertainty. Numerous theoretical and practical problems have been solved using this approach. Scholars have shown sustained interest in the theory's core concepts and operations since its inception. In this study, we propose the binary extended theta operation, a special binary soft set operation, and provide a thorough analysis of its basic algebraic features. We also study the distribution of this operation over certain types of binary soft set operations. By considering its algebraic properties and distribution rules, we show that, when combined with specific binary soft set operations, the binary extended theta operation forms many important algebraic structures within the collection of binary soft sets over the universe under certain conditions. The fundamental conceptual difference between the proposed binary extended theta operation and existing binary extended operations in the literature is that unlike approaches based on positive information aggregation, the theta operation systematically extracts negative information through common parameters and offers a unique and complementary tool, particularly for decision problems requiring reliable elimination, risk exclusion, and error detection. Further applications, including cryptology and decision-making, rely on operations of binary soft sets, making this theoretical subject essential from both theoretical and practical perspectives.

Keywords: Binary soft sets, Binary soft set operations, Binary extended theta operation, Semiring

1. Introduction

Soft set theory has become a powerful mathematical tool for handling uncertainty since its introduction by Molodtsov [1] in 1999. Maji et al. [2], Pei and Miao [3], and Ali et al. [4,5] are the pioneers in soft set operations and developed the algebraic basis of soft set theory. The uni-int decision-making method for soft sets was proposed by Çağman and Enginoğlu [6].

Soft int-groups were introduced by Çağman et al. [7], while Sezer et al. [8] introduced soft N-groups to further

generalize group theory within the framework of soft sets. Later, Sezer et al. [9] expanded on this by introducing the notion of N-group SI-actions. Atagün and Sezer [10] explored soft semimodules, establishing connections between module theory and soft sets. A notable shift towards applications began with Khan et al. [11], who characterized Abel Grassmann's groupoids via soft ideals. Atagün and Sezgin [12] analyzed soft substructures of groups and semirings. Around the same time, soft sets were applied to near-ring theory [13] and ideal theory [14], further expanding the theory's domain. Atagün et al. [15] explored

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P-properties in near-rings, while additional work on prime, maximal, and principal soft ideals of rings and soft covered ideals of semigroups were discussed in [16,17]. In addition, the social relevance of soft set theory has been highlighted in emerging fields such as social networks, as explored by Riaz et al. [18]. Soft intersection almost subsemigroups and ideals of semigroups were addressed in [19,20]. Finally, the most recent contribution presents a complete study of the star-product and-product of soft sets [21,22].

Since its introduction by Açıkgöz and Taş [23], the concept of the binary soft set (BSS) has attracted considerable interest among decision scientists and mathematicians. By linking binary relations to components of the parameter set, this novel theory extends Molodtsov's soft set theory and offers a more adaptable framework for handling uncertainty in complicated settings.

Binary soft topological spaces [24-27], soft subspaces [28], vague BSSs [29], separation axioms, [30-32], weak binary soft structures [33], bipolar and fuzzy binary soft models [34-36], and binary soft functions [37] are foundational studies.

The model's potential in expert systems, artificial intelligence, and advanced decision theory has been demonstrated by various recent contributions [38-40]. In fields like healthcare [38] and multi-agent systems [41], soft mappings and BSS-based decision processes have shown promise. Besides, extensions into neutrosophic structures and hypersoft sets showed how the theory can be used to complex logical systems [39,42-44]. Integration of BSSs with fuzzy and intuitionistic fuzzy frameworks [45], neutrosophic systems [42,46], and plithogenic models [41] have become a significant trend. Additionally, several studies explore binary soft α -open sets [47], soft θ -open and ω -open sets [48], soft $\omega\beta$ -open sets [49], and soft homogeneous components [50], enhancing the structural analysis of binary soft topologies.

Contributions have also focused on extending classical mathematical results to BSSs, such as selection principles [51], object interaction sets [52], and applications in game theory [53]. Meanwhile, binary soft closure spaces and limit points have been reconsidered in recent studies to refine the notions of continuity and compactness [31,54]. The ongoing development of BSS theory is further evidenced by continued work on structural properties, mappings [30,32], topological functions, fuzzy BSSs and their applications, fuzzy parameterized fuzzy BSSs and their operations, binary hypersoft sets, and fuzzy binary soft topological spaces [55-59]. Dalkılıç [60] contributed by introducing and thoroughly developing the concepts of relations on BSSs, blocks, partitions, compositions, and binary soft functions.

The concept of BSSs was first introduced by Açıkgöz and Taş [23]. Soylu [61] revisited the foundational concepts of BSS theory, and updated existing definitions as necessary. Additionally, many new concepts regarding the BSS such as empty BSS, absolute BSS, and relative null/whole BSS with respect to a fixed parameter set, were

introduced. Subsequently, different types of operations on BSSs, such as binary restricted operations and binary extended operations were presented.

The vital role that BSSs plays in contemporary mathematics and decision analysis is confirmed by this extensive body of literature. By investigating the binary extended theta operation of BSSs, the current study builds on this foundation and attempts to advance the theory of BSSs by offering both theoretical insights and practical applications aligned with current research. The primary motivation for this study is that existing binary extended operations defined in the binary soft set literature are largely focused on positive information integration. In particular, operations such as extended union and extended intersection aim to combine acceptable or positive information from different sources. In contrast, a critical element in many real-world applications is determining which elements are universally rejected by all sources, i.e., identifying elements that must be definitively eliminated. In this context, the binary extended theta operation proposed in this study, rather than producing positive consensus, reveals common negative information by taking the intersection of the complements of binary soft sets on common parameters. Thus, the theta operation conceptually differs from existing binary extended operations in the literature and provides a new algebraic perspective based on negative information for binary soft set theory. By incorporating a real-life micro-example into the study, it has been clearly demonstrated that the theta process plays a practical and meaningful role, particularly in scenarios involving quality control, risk analysis, and reliable elimination, in contrast to classical binary extended processes. We demonstrate that the binary extended theta operation produces numerous significant algebraic structures within the collection of soft sets over the universe under certain conditions, by considering its algebraic properties and distribution principles. The structure of this study is as follows: The fundamental concepts of BSSs are reviewed in Section 2. In section 3, the binary extended theta operation, is defined and its algebraic properties are thoroughly examined together with its distributions over other BSS operations. A thorough examination of the algebraic structures produced in the set of BSSs over the universe by these operations is provided. In the conclusion section, we discuss the significance of the study's findings and their potential applications.

2. Preliminaries

The basic concepts of the BSSs are provided in this section.

Definition 2.1 [23] Let U_1 and U_2 be initial universe sets, E be the set of all parameters characterizing the elements of the initial universal sets, $P(U_1)$ and $P(U_2)$ be the power sets of U_1 and U_2 , respectively and $A \subseteq E$. A binary soft set (BSS) (F^A, A) over U_1, U_2 is defined by the set of ordered pairs

$(F^A, A) = \{(e, (X, Y)) : e \in A, F^A(e) = (X, Y), X \subseteq U_1, Y \subseteq U_2\}$
 where

$$F^A: A \rightarrow P(U_1) \times P(U_2)$$

is a function such that $F^A(e) = (X, Y)$, for each $e \in A$, $X \subseteq U_1$ and $Y \subseteq U_2$.

For the sake brevity, throughout this study, the BSS (F^A, A) will be denoted as F^A . The set of all BSSs over U_1, U_2 will be denoted by $S^E(U_1, U_2)$. Additionally, the set of all BSSs over U_1, U_2 with the fixed parameter subset A of subset E will be denoted by $S^A(U_1, U_2)$.

Definition 2.2 [23,61] Let F^A be a BSS over U_1, U_2 . If $F^A(a) = (\emptyset, \emptyset)$ for each $a \in A$, then F^A is called a relative null BSS with respect to A , and it is denoted by \emptyset^A . If $F^E(a) = (\emptyset, \emptyset)$ for each $a \in E$, then F^E is called relative null BSS with respect to E , and it is denoted by \emptyset^E .

A BSS with an empty parameter set, that is, $F: \emptyset \rightarrow P(U_1) \times P(U_2)$ is called an empty BSS and is denoted by \emptyset^\emptyset .

Definition 2.3 [23,61] Let F^A be a BSS over U_1, U_2 . If $F^A(a) = (U_1, U_2)$ for each $a \in A$, then F^A is called a relative whole BSS with respect to A , and it is denoted by \mathcal{U}^A . The relative whole BSS \mathcal{U}^E with respect to E is called the absolute BSS over U_1, U_2 .

Definition 2.4 [23] Let F^A and G^B be BSSs over U_1, U_2 . F^A is called a binary soft subset of G^B if

i) $A \subseteq B$

ii) $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$, where $F^A(e) = (X_1, Y_1)$ and $G^B(e) = (X_2, Y_2)$, for each $e \in A$. It is denoted by $F^A \tilde{\subseteq} G^B$.

F^A is called a binary soft super set of G^B , if G^B is a binary soft subset of F^A , and it is denoted by $F^A \tilde{\supseteq} G^B$. F^A is called binary soft equal to G^B , if $F^A \tilde{\subseteq} G^B$ and $G^B \tilde{\subseteq} F^A$.

Throughout this study, for each $e \in A$, where $F^A(e) = (X_1, Y_1)$ and $G^A(e) = (X_2, Y_2)$, it is meant by $F^A(e) \subseteq G^A(e)$, namely $(X_1, Y_1) \subseteq (X_2, Y_2)$ that $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$.

Definition 2.5 [61] Let F^A be a BSS over U_1, U_2 . The relative complement of F^A , denoted by $(F^A)^c$, is a function

$$(F^A)^c: A \rightarrow P(U_1) \times P(U_2)$$

given by $(F^A)^c(e) = (U_1 - X, U_2 - Y) = (X', Y')$ for each $e \in A$ such that $F^A(e) = (X, Y)$.

Definition 2.6 [23,61] Let F^A and G^B be BSSs over U_1, U_2 . The binary restricted intersection (union) of F^A and G^B is the BSS H^C , denoted by $F^A \tilde{\cap}_R G^B = H^C$ ($F^A \tilde{\cup}_R G^B = H^C$), where $C = A \cap B$, and $H^C(c) = ((F^A(c))_1 \cap (G^B(c))_1, (F^A(c))_2 \cap (G^B(c))_2)$, ($H^C(c) = ((F^A(c))_1 \cup (G^B(c))_1, (F^A(c))_2 \cup (G^B(c))_2)$), for each $c \in C$. Here, if $C = A \cap B = \emptyset$, then $F^A \tilde{\cap}_R G^B = H^C = \emptyset^\emptyset$ ($F^A \tilde{\cup}_R G^B = H^C = \emptyset^\emptyset$).

Definition 2.7 [61] Let F^A and G^B be BSSs over U_1, U_2 . The binary extended intersection of F^A and G^B is the BSS H^C , denoted by $F^A \tilde{\cap}_E G^B = H^C$, where $C = A \cup B$, and for each $c \in C$,

$$H^C(c) = \begin{cases} ((F^A(c))_1, (F^A(c))_2) & c \in A - B \\ ((G^B(c))_1, (G^B(c))_2) & c \in B - A \\ ((F^A(c))_1 \cap (G^B(c))_1, (F^A(c))_2 \cap (G^B(c))_2) & c \in A \cap B \end{cases}$$

Definition 2.8 [23] Let F^A and G^B be BSSs over U_1, U_2 . The binary extended union of F^A and G^B is the BSS H^C , denoted by $F^A \tilde{\cup}_E G^B = H^C$, where $C = A \cup B$, and for each $c \in C$,

$$H^C(c) = \begin{cases} ((F^A(c))_1, (F^A(c))_2) & c \in A - B \\ ((G^B(c))_1, (G^B(c))_2) & c \in B - A \\ ((F^A(c))_1 \cup (G^B(c))_1, (F^A(c))_2 \cup (G^B(c))_2) & c \in A \cap B \end{cases}$$

Definition 2.9 [61] Let F^A and G^B be BSSs over U_1, U_2 . The binary intersection of F^A and G^B is the BSS H^A , denoted by $F^A \tilde{\cap} G^B = H^A$, and for each $a \in A$,

$$H^A(a) = \begin{cases} ((F^A(a))_1, (F^A(a))_2) & a \in A - B \\ ((F^A(a))_1 \cap (G^B(a))_1, (F^A(a))_2 \cap (G^B(a))_2) & a \in A \cap B \end{cases}$$

Definition 2.10 [61] Let F^A and G^B be BSSs over U_1, U_2 . The binary union of F^A and G^B is the BSS H^A , denoted by $F^A \tilde{\cup} G^B = H^A$, and for each $a \in A$,

$$H^A(c) = \begin{cases} ((F^A(a))_1, (F^A(a))_2) & a \in A - B \\ (((F^A(a))_1 \cup (G^B(a))_1, (F^A(a))_2 \cup (G^B(a))_2) & a \in A \cap B \end{cases}$$

3. Binary extended theta operation of binary soft sets

3.1 Binary extended theta operation with its fundamental properties

In this subsection, the definition of the binary extended theta operation together with its properties are provided with detailed proofs.

Definition 3.1.1 Let F^W and G^W be BSSs over U_1, U_2 such that $F^W = (\mathfrak{S}_1, \mathfrak{S}_2)$ for each $w \in W$ and $G^W(w) = (\mathcal{L}_1, \mathcal{L}_2)$ for each $w \in W$. The binary extended theta of F^W and G^W is the BSS H^C , denoted by $F^W \tilde{\theta}_E G^W = H^C$, where $C = W \cup W$ and for each $c \in C$.

$$H^C(c) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & c \in W - W \\ (\mathcal{L}_1, \mathcal{L}_2) & c \in W - W \\ ((\mathfrak{S}_1 \theta \mathcal{L}_1, \mathfrak{S}_2 \theta \mathcal{L}_2) & c \in W \cap W \end{cases}$$

Here, $(\mathfrak{S}_1 \theta \mathcal{L}_1, \mathfrak{S}_2 \theta \mathcal{L}_2) = (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2')$.

For more on theta (θ) operation of sets, we refer to [62]. By Definition 3.1.1, if $W = \emptyset$, then $F^W \tilde{\theta}_E G^W = G^W$, if $W = \emptyset$, then $F^W \tilde{\theta}_E G^W = G^W$, and if $W = W = \emptyset$, then $F^W \tilde{\theta}_E G^W = \emptyset^\emptyset$.

Example 3.1.2 Let $U_1 = \{t_1, t_2, t_3, t_4, t_5\}$, $U_2 = \{h_1, h_2, h_3, h_4\}$, $E = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$, $W = \{\psi_1, \psi_2, \psi_4, \psi_5\}$, $Y = \{\psi_1, \psi_3, \psi_4, \psi_5\}$ and the BSSs F^W and G^W over U_1, U_2 be defined as follows:

$$F^W = \{(\psi_1, (\{t_1, t_2, t_3\}, \{h_1, h_2\})), (\psi_2, (\{t_1, t_4, t_5\}, \{h_2, h_3\})), (\psi_4, (\{t_2, t_5\}, \{h_1, h_2, h_3\})), (\psi_5, (\{t_2, t_3\}, \{h_1, h_4\}))\}$$

$$G^W = \{(\psi_1, (\{t_1, t_3, t_4\}, \{h_1, h_2, h_4\})), (\psi_3, (\{t_1, t_5\}, \{h_2, h_3\})), (\psi_4, (U_1, \{h_2, h_3, h_4\})), (\psi_5, (\{t_3, t_4\}, \{h_2, h_3, h_4\}))\}$$

Let $F^W \tilde{\theta}_\epsilon G^W = H^{W \cup Y}$, where $W \cup Y = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$. Since $W - Y = \{\psi_2\}$, $Y - W = \{\psi_3\}$, and $W \cap Y = \{\psi_4, \psi_5\}$,

$$F^W \tilde{\theta}_\epsilon G^W = \{(\psi_1, (\{t_5\}, \{h_3\})), (\psi_2, (\{t_1, t_4, t_5\}, \{h_2, h_3\})), (\psi_3, (\{t_1, t_5\}, \{h_2, h_3\})), (\psi_4, (\emptyset, \emptyset)), (\psi_5, (\{t_1, t_5\}, \emptyset))\}$$

In this example, the fact that some components become empty sets is related to parameter differences and the combinatorial structure of the theta operation. For example, in the operation $F^W \tilde{\theta}_\epsilon G^W$, since the union of $F^W(\psi_4)$ and $G^W(\psi_4)$ is the universal set, the result of the operation is the empty set (\emptyset) for ψ_4 . This behavior reflects the self-negation and parameter-based union properties of the theta operation and is not observed in classical union/intersection operations. Therefore, the example clearly demonstrates the algebraic logic of the theta operation and the effect of parameter differences on the result.

Example 3.1.3 (A Real-Life Quality Control Problem) Quality control of products manufactured in a factory is performed by two different inspectors. There are two separate product groups in the factory, each from a different production lines and tested independently. Products from the first production line form the first product group, while products from the second production line form the second product group. These two product groups undergo various quality tests using different testing equipment.

Inspector A is responsible for mechanical and electrical tests, while Inspector B is responsible for electrical and safety tests. Therefore, some tests are evaluated by only one inspector, while others are evaluated by both inspectors. The goal is to evaluate the reports of these two inspectors together to reliably identify both successful products for the two production lines and, in particular, defective products that must be eliminated.

Products on the first production line:

$$U_1 = \{t_1, t_2, t_3, t_4\},$$

Products on the second production line:

$$U_2 = \{h_1, h_2, h_3, h_4\}$$

Test types (set of parameters):

$$E = \{\psi_1, \psi_2, \psi_3\}$$

Here ψ_1 : Mechanical test, ψ_2 : Electrical test, ψ_3 : Safety test.

Tests evaluated by Inspector A:

$$W = \{\psi_1, \psi_2\},$$

Tests evaluated by Inspector B:

$$Y = \{\psi_2, \psi_3\}$$

and the BSSs F^W and G^W over U_1, U_2 be defined as follows:

$$F^W = \{(\psi_1, (\{t_1, t_2\}, \{h_1\})), (\psi_2, (\{t_2, t_3\}, \{h_2\}))\}$$

According to Inspector A, during the mechanical test, products numbered t_1 and t_2 from the first production line and product numbered h_1 from the second production line were found to be successful. In the safety test, products numbered t_2 and t_3 from the first production line and product numbered h_2 from the second production line were found to be successful.

$$G^W = \{(\psi_2, (\{t_1, t_3\}, \{h_2, h_3\})), (\psi_3, (\{t_3\}, \{h_3\}))\}$$

According to Inspector B, during the electrical test, products numbered t_1 and t_3 from the first production line and products numbered h_2 and h_3 from the second production line were found to be successful. In the safety test, product numbered t_3 from the first production line and product numbered h_3 from the second production line were found to be successful.

Let $F^W \tilde{\theta}_\epsilon G^W = H^{W \cup Y}$, where $W \cup Y = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$. Since $W - Y = \{\psi_1\}$, $Y - W = \{\psi_3\}$, and $W \cap Y = \{\psi_2\}$, this means that the mechanical test is evaluated only by Inspector A, the safety test is evaluated only by Inspector B, and the electrical test is evaluated by both inspectors.

$$F^W \tilde{\theta}_\epsilon G^W = \{(\psi_1, (\{t_1, t_2\}, \{h_1\})), (\psi_2, (\{t_4\}, \{h_1, h_4\})), (\psi_3, (\{t_3\}, \{h_3\}))\}$$

This result indicates that, according to both inspectors in the electrical test, product numbered t_4 from the first production line; products numbered h_1 and h_4 from the second production line has definitely failed and should be safely discarded.

Additionally, since the mechanical test $\psi_1 \in W - Y$ is evaluated only by Inspector A,

$$(F^W \tilde{\theta}_\epsilon G^W)(\psi_1) = (\{t_1, t_2\}, \{h_1\}),$$

and since security test $\psi_3 \in \mathcal{W} - \mathcal{W}$ is evaluated only by Inspector B,

$$(F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{W}})(\psi_3) = (\{t_3\}, \{h_3\}),$$

is obtained. Interpretation: The binary extended theta operation preserves existing information in tests reviewed

by a single expert, but in tests reviewed jointly by two experts, it produces a common negative agreement rather than a positive agreement, thereby reliably identifying products that will definitely be rejected in multi-criteria quality control systems as it is presented in Table 1.

Table 1. Binary soft set operations: Focus and interpretation

Operation	Focus	Interpretation
Binary Extended Union	Positive acceptance	Any accepted by either source
Binary Extended Intersection	Common acceptance	Accepted by both sources
Binary Extended Theta (proposed)	Common rejection	Rejected by both sources

Note 3.1.4 Binary restricted and binary extended intersection operations coincide in $S^A(U_1, U_2)$. That is, $F^A \tilde{\theta}_R G^A = F^A \tilde{\theta}_\epsilon G^A$.

Proposition 3.1.5 $\tilde{\theta}_\epsilon$ is closed in $S^E(U_1, U_2)$.

Proof: It is evident that $\tilde{\theta}_\epsilon$ is a binary operation in $S^E(U_1, U_2)$. That is,

$$\tilde{\theta}_\epsilon : S^E(U_1, U_2) \times S^E(U_1, U_2) \rightarrow S^E(U_1, U_2)$$

$$(F^{\mathcal{V}}, G^{\mathcal{W}}) \rightarrow F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{W}} = H^{\mathcal{V} \cup \mathcal{W}}$$

Namely, if $F^{\mathcal{V}}$ and $G^{\mathcal{W}}$ are BSSs over U_1, U_2 then so is $F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{W}}$. Similarly, let \mathcal{W} be a fixed subset of E , then

$$\tilde{\theta}_\epsilon : S^{\mathcal{V}}(U_1, U_2) \times S^{\mathcal{W}}(U_1, U_2) \rightarrow S^{\mathcal{V}}(U_1, U_2)$$

$$(F^{\mathcal{V}}, G^{\mathcal{W}}) \rightarrow F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{W}} = H^{\mathcal{V} \cup \mathcal{W}} = H^{\mathcal{V}}$$

Namely, if $F^{\mathcal{V}}$ and $G^{\mathcal{W}}$ are BSSs over U_1, U_2 then so is $F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{W}}$.

Proposition 3.1.6 Let $F^{\mathcal{V}}, G^{\mathcal{W}}$ and $H^{\mathcal{Q}}$ be BSSs over U_1, U_2 . Then, $(F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{W}}) \tilde{\theta}_\epsilon H^{\mathcal{Q}} = F^{\mathcal{V}} \tilde{\theta}_\epsilon (G^{\mathcal{W}} \tilde{\theta}_\epsilon H^{\mathcal{Q}})$, where $\mathcal{W} \cap \mathcal{W} \cap \mathcal{Q} = \emptyset$.

Proof: Let $F^{\mathcal{V}}(\psi) = (\mathfrak{S}_1, \mathfrak{S}_2)$ for each $\psi \in \mathcal{V}$, $G^{\mathcal{W}}(\omega) = (\mathcal{L}_1, \mathcal{L}_2)$ for each $\omega \in \mathcal{W}$, $H^{\mathcal{Q}}(\varrho) = (\mathcal{B}_1, \mathcal{B}_2)$ for each $\varrho \in \mathcal{Q}$, and $F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{W}} = T^{\mathcal{V} \cup \mathcal{W}}$, where

$$T^{\mathcal{V} \cup \mathcal{W}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - \mathcal{W} \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{W} - \mathcal{W} \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in \mathcal{W} \cap \mathcal{W} \end{cases}$$

for each $\lambda \in \mathcal{W} \cup \mathcal{W}$. Let $T^{\mathcal{V} \cup \mathcal{W}} \tilde{\theta}_\epsilon H^{\mathcal{Q}} = S^{(\mathcal{V} \cup \mathcal{W}) \cup \mathcal{Q}}$, where

$$S^{(\mathcal{V} \cup \mathcal{W}) \cup \mathcal{Q}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (\mathcal{W} - \mathcal{W}) - \mathcal{Q} = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in (\mathcal{W} - \mathcal{W}) - \mathcal{Q} = \mathcal{W}' \cap \mathcal{W} \cap \mathcal{Q}' \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in (\mathcal{W} \cap \mathcal{W}) - \mathcal{Q} = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in \mathcal{Q} - (\mathcal{W} \cup \mathcal{W}) = \mathcal{W}' \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') & \lambda \in (\mathcal{W} - \mathcal{W}) \cap \mathcal{Q} = \mathcal{W}' \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathcal{L}_1' \cap \mathcal{B}_1', \mathcal{L}_2' \cap \mathcal{B}_2') & \lambda \in (\mathcal{W} - \mathcal{W}) \cap \mathcal{Q} = \mathcal{W}' \cap \mathcal{W} \cap \mathcal{Q}' \\ ((\mathfrak{S}_1 \cup \mathcal{L}_1) \cap \mathcal{B}_1'), ((\mathfrak{S}_2 \cup \mathcal{L}_2) \cap \mathcal{B}_2') & \lambda \in (\mathcal{W} \cap \mathcal{W}) \cap \mathcal{Q} = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \end{cases}$$

for each $\lambda \in (\mathcal{W} \cup \mathcal{W}) \cup \mathcal{Q}$.

Assume that $G^{\mathcal{W}} \tilde{\theta}_\epsilon H^{\mathcal{Q}} = R^{\mathcal{W} \cup \mathcal{Q}}$, where

$$R^{\mathcal{W} \cup \mathcal{Q}} = \begin{cases} (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{W} - \mathcal{Q} \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in \mathcal{Q} - \mathcal{W} \\ (\mathcal{L}_1' \cap \mathcal{B}_1', \mathcal{L}_2' \cap \mathcal{B}_2') & \lambda \in \mathcal{W} \cap \mathcal{Q} \end{cases}$$

for each $\lambda \in \mathcal{W} \cup \mathcal{Q}$ and $F^{\mathcal{V}} \tilde{\theta}_\epsilon R^{\mathcal{W} \cup \mathcal{Q}} = K^{\mathcal{V} \cup (\mathcal{W} \cup \mathcal{Q})}$, where

$$K^{\mathcal{V} \cup (\mathcal{W} \cup \mathcal{Q})}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - (\mathcal{W} \cup \mathcal{Q}) = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in (\mathcal{W} - \mathcal{Q}) - \mathcal{W} = \mathcal{W}' \cap \mathcal{W} \cap \mathcal{Q}' \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in (\mathcal{Q} - \mathcal{W}) - \mathcal{W} = \mathcal{W}' \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathcal{L}_1' \cap \mathcal{B}_1', \mathcal{L}_2' \cap \mathcal{B}_2') & \lambda \in (\mathcal{W} \cap \mathcal{Q}) - \mathcal{W} = \mathcal{W}' \cap \mathcal{W} \cap \mathcal{Q}' \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in \mathcal{W} \cap (\mathcal{W} - \mathcal{Q}) = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') & \lambda \in \mathcal{W} \cap (\mathcal{Q} - \mathcal{W}) = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \\ ((\mathfrak{S}_1' \cap (\mathcal{L}_1 \cup \mathcal{B}_1)), (\mathfrak{S}_2' \cap (\mathcal{L}_2 \cup \mathcal{B}_2))) & \lambda \in (\mathcal{W} \cap \mathcal{W}) \cap \mathcal{Q} = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \end{cases}$$

for each $\lambda \in \mathcal{V} \cup (\mathcal{W} \cup \mathcal{Q})$. Thus, $S^{(\mathcal{V} \cup \mathcal{W}) \cup \mathcal{Q}} = K^{\mathcal{V} \cup (\mathcal{W} \cup \mathcal{Q})}$, where $\mathcal{W} \cap \mathcal{W} \cap \mathcal{Q} = \emptyset$.

From a different point of view, $((\mathfrak{S}_1 \cup \mathcal{L}_1) \cap \mathcal{B}_1')$, $((\mathfrak{S}_2 \cup \mathcal{L}_2) \cap \mathcal{B}_2')$ $= ((\mathfrak{S}_1' \cap (\mathcal{L}_1 \cup \mathcal{B}_1)), \mathfrak{S}_2' \cap (\mathcal{L}_2 \cup \mathcal{B}_2))$ holds in the unique non-degenerate meaningful case when $\mathfrak{S}_1 = \mathcal{B}_1$ and $\mathfrak{S}_2 = \mathcal{B}_2$, namely, $F^{\mathcal{V}} = H^{\mathcal{Q}}$.

That is, $\tilde{\theta}_\epsilon$ is associative in $S^E(U_1, U_2)$ under certain conditions (UCC).

Note 3.1.7 The conditions necessary to achieve

associativity are natural from a mathematical perspective and can be found in both theoretical models and practical applications. In particular, such equality can occur when certain evaluation criteria are equivalent or when different experts make similar classifications based on the same parameters. Therefore, equality is not unique to abstract or artificial scenarios but can arise meaningfully in specific application contexts.

Remark 3.1.8 Let $F^{\mathcal{V}}, G^{\mathcal{V}}$ and $H^{\mathcal{V}}$ be BSSs over U_1, U_2 . Then, $(F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{V}}) \tilde{\theta}_\epsilon H^{\mathcal{V}} \neq F^{\mathcal{V}} \tilde{\theta}_\epsilon (G^{\mathcal{V}} \tilde{\theta}_\epsilon H^{\mathcal{V}})$.

Proof: The proof follows from Proposition 3.1.7. That is, $\tilde{\theta}_\epsilon$ is not associative in $S^{\mathcal{V}}(U_1, U_2)$.

Proposition 3.1.9 Let $F^{\mathcal{V}}$ and $G^{\mathcal{W}}$ be BSSs over U_1, U_2 . Then, $F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{W}} = G^{\mathcal{W}} \tilde{\theta}_\epsilon F^{\mathcal{V}}$.

Proof: Let $F^{\mathcal{W}}(\mathcal{Y}) = (\mathfrak{S}_1, \mathfrak{S}_2)$ for each $\mathcal{Y} \in \mathcal{W}$, $G^{\mathcal{W}}(\mathcal{Y}) = (\mathcal{L}_1, \mathcal{L}_2)$ for each $\mathcal{Y} \in \mathcal{W}$, and $F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{W}} = T^{\mathcal{V} \cup \mathcal{W}}$, where

$$T^{\mathcal{V} \cup \mathcal{W}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - \mathcal{W} \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{W} - \mathcal{V} \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in \mathcal{W} \cap \mathcal{V} \end{cases}$$

for each $\lambda \in \mathcal{W} \cup \mathcal{W}$. Let $G^{\mathcal{W}} \tilde{\theta}_\epsilon F^{\mathcal{V}} = R^{\mathcal{W} \cup \mathcal{V}}$, where

$$R^{\mathcal{W} \cup \mathcal{V}} = \begin{cases} (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{W} - \mathcal{W} \\ (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - \mathcal{V} \\ (\mathcal{L}_1' \cap \mathfrak{S}_1', \mathcal{L}_2' \cap \mathfrak{S}_2') & \lambda \in \mathcal{W} \cap \mathcal{V} \end{cases}$$

for each $\lambda \in \mathcal{W} \cup \mathcal{W}$. Thereby, $T^{\mathcal{V} \cup \mathcal{W}} = R^{\mathcal{W} \cup \mathcal{V}}$. That is, $\tilde{\theta}_\epsilon$ is commutative in $S^{\mathcal{E}}(U_1, U_2)$. It is also obvious that $F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{W}} = G^{\mathcal{W}} \tilde{\theta}_\epsilon F^{\mathcal{V}}$, namely, $\tilde{\theta}_\epsilon$ is commutative in $S^{\mathcal{V}}(U_1, U_2)$ as well.

Theorem 3.1.10 Let $F^{\mathcal{V}}$ be a BSS over U_1, U_2 . Then, $F^{\mathcal{V}} \tilde{\theta}_\epsilon F^{\mathcal{V}} = (F^{\mathcal{V}})^c$.

Proof: Let $F^{\mathcal{V}}(\mathcal{Y}) = (\mathfrak{S}_1, \mathfrak{S}_2)$ for each $\mathcal{Y} \in \mathcal{W}$, and $F^{\mathcal{V}} \tilde{\theta}_\epsilon F^{\mathcal{V}} = T^{\mathcal{V} \cup \mathcal{V}} = T^{\mathcal{V}}$, where

$$T^{\mathcal{V}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - \mathcal{W} = \emptyset \\ (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - \mathcal{V} = \emptyset \\ (\mathfrak{S}_1' \cap \mathfrak{S}_1', \mathfrak{S}_2' \cap \mathfrak{S}_2') & \lambda \in \mathcal{W} \cap \mathcal{V} = \mathcal{W} \end{cases}$$

Then, $T^{\mathcal{V}}(\lambda) = (\mathfrak{S}_1', \mathfrak{S}_2')$ for each $\lambda \in \mathcal{W}$, implying that $T^{\mathcal{V}} = (F^{\mathcal{V}})^c$. Namely, $\tilde{\theta}_\epsilon$ is not idempotent in $S^{\mathcal{E}}(U_1, U_2)$.

Note 3.1.11 This equality demonstrates that the proposed binary extended theta operation naturally generalizes the complement operation. In particular, when applied to identical inputs, the theta operation behaves like a unary negation operator. This property demonstrates the internal consistency of the proposed operation and confirms that the theta operation captures a type of self-negation under common rejection, which is not observed in existing binary soft operations such as extended union or extended intersection. This makes the operation more than a standard

binary operation as it is a multi-layered algebraic structure that incorporates both binary and unary information. In particular, due to the single operator (complement) it contains, it systematically reveals the potential for self-negation not present in classical binary soft operations. This feature reinforces the flexibility and internal consistency of the theta operation, strengthening the value and general applicability of the proposed algebraic structure.

Proposition 3.1.12 Let $F^{\mathcal{V}}$ be a BSS over U_1, U_2 . Then, $F^{\mathcal{V}} \tilde{\theta}_\epsilon \emptyset^{\mathcal{V}} = \emptyset^{\mathcal{V}} \tilde{\theta}_\epsilon F^{\mathcal{V}} = F^{\mathcal{V}}$.

Proof: Let $F^{\mathcal{V}}(\mathcal{Y}) = (\mathfrak{S}_1, \mathfrak{S}_2)$ for each $\mathcal{Y} \in \mathcal{W}$, $\emptyset^{\mathcal{V}} = K^{\mathcal{V}}$ and $K^{\mathcal{V}}(\mathcal{Y}) = (\mathcal{L}_1, \mathcal{L}_2)$. Assume that $F^{\mathcal{V}} \tilde{\theta}_\epsilon K^{\mathcal{V}} = T^{\mathcal{V} \cup \emptyset} = T^{\mathcal{V}}$, where

$$T^{\mathcal{V}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - \emptyset = \mathcal{W} \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \emptyset - \mathcal{W} = \emptyset \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in \mathcal{W} \cap \emptyset = \emptyset \end{cases}$$

for each $\lambda \in \mathcal{W}$. Thus, $T^{\mathcal{V}}(\lambda) = (\mathfrak{S}_1, \mathfrak{S}_2)$, implying that $T^{\mathcal{V}} = F^{\mathcal{V}}$.

Note 3.1.13 According to Proposition 3.1.12, the identity element of $\tilde{\theta}_\epsilon$ is the BSS $\emptyset^{\mathcal{V}}$ in $S^{\mathcal{E}}(U_1, U_2)$. Here, note that for $G^{\mathcal{W}} \in S^{\mathcal{E}}(U_1, U_2)$, there does not exist $F^{\mathcal{V}} \in S^{\mathcal{E}}(U_1, U_2)$ such that $F^{\mathcal{V}} \tilde{\theta}_\epsilon G^{\mathcal{W}} = G^{\mathcal{W}} \tilde{\theta}_\epsilon F^{\mathcal{V}} = \emptyset^{\mathcal{V}}$, because this situation requires $\mathcal{W} \cup \mathcal{W} = \emptyset$, that is $\mathcal{W} = \emptyset$ and $\mathcal{W} = \emptyset$, that is, this requires $F^{\mathcal{V}} = G^{\mathcal{W}} = \emptyset^{\mathcal{V}}$. From this, we conclude that in the set $S^{\mathcal{E}}(U_1, U_2)$, there is no inverse element for any element other than the BSS $\emptyset^{\mathcal{V}}$ with respect to $\tilde{\theta}_\epsilon$. Naturally, the inverse of $\emptyset^{\mathcal{V}}$, the identity element of $S^{\mathcal{E}}(U_1, U_2)$, is itself, i.e. $\emptyset^{\mathcal{V}}$.

Theorem 3.1.14 $(S^{\mathcal{E}}(U_1, U_2), \tilde{\theta}_\epsilon)$ is a commutative, not idempotent monoid, whose identity is $\emptyset^{\mathcal{V}}$ UCCs.

Proof: Using Proposition 3.1.5, Proposition 3.1.6, Proposition 3.1.9, Theorem 3.1.10, Proposition 3.1.12 and Note 3.1.13, the proof is clear.

Note 3.1.15 The theta operation in the classical set is not associative [62], and therefore does not form a monoid. In the context of binary soft sets, we can impose certain parameter conditions that make the operation associative—these conditions are not overly restrictive. Under these conditions, the binary extended theta operation is not only associative, but also enables the creation of meaningful binary algebraic structures in $(S^{\mathcal{E}}(U_1, U_2))$. This demonstrates that the operation has an independent algebraic value beyond classical set theory.

Proposition 3.1.16 Let $F^{\mathcal{V}}$ be a BSS over U_1, U_2 . Then, $F^{\mathcal{V}} \tilde{\theta}_\epsilon \emptyset^{\mathcal{V}} = \emptyset^{\mathcal{V}} \tilde{\theta}_\epsilon F^{\mathcal{V}} = (F^{\mathcal{V}})^c$.

Proof: Let $\emptyset^{\mathcal{V}} = K^{\mathcal{V}}$ and $F^{\mathcal{V}}(\mathcal{Y}) = (\mathfrak{S}_1, \mathfrak{S}_2)$ for each $\mathcal{Y} \in \mathcal{W}$. Then, $K^{\mathcal{V}}(\mathcal{Y}) = (\emptyset, \emptyset)$ for each $\mathcal{Y} \in \mathcal{W}$. Assume that $F^{\mathcal{V}} \tilde{\theta}_\epsilon K^{\mathcal{V}} = T^{\mathcal{V} \cup \emptyset} = T^{\mathcal{V}}$, where

$$T^{\mathcal{V}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - \emptyset = \mathcal{W} \\ (\emptyset, \emptyset) & \lambda \in \emptyset - \mathcal{W} = \emptyset \\ (\mathfrak{S}_1' \cap \emptyset', \mathfrak{S}_2' \cap \emptyset') & \lambda \in \mathcal{W} \cap \emptyset = \emptyset \end{cases}$$

for each $\lambda \in \mathcal{W}$. Hence,

$$T^{\mathcal{V}}(\lambda) = (\mathfrak{S}_1' \cap U_1, \mathfrak{S}_2' \cap U_2) = (\mathfrak{S}_1', \mathfrak{S}_2') = (F^{\mathcal{V}})^r(\lambda),$$

implying that $T^{\mathcal{V}} = (F^{\mathcal{V}})^r$.

Proposition 3.1.17 Let $F^{\mathcal{V}}$ be a BSS over U_1, U_2 . Then, $F^{\mathcal{V}} \tilde{\theta}_\varepsilon \mathcal{U}^{\mathcal{V}} = \mathcal{U}^{\mathcal{V}} \tilde{\theta}_\varepsilon F^{\mathcal{V}} = \emptyset^{\mathcal{V}}$.

Proof: Let $\mathcal{U}^{\mathcal{V}} = K^{\mathcal{V}}$ and $F^{\mathcal{V}}(\psi) = (\mathfrak{S}_1', \mathfrak{S}_2')$ for each $\psi \in \mathcal{W}$. Then, $K^{\mathcal{V}}(\psi) = (U_1, U_2)$ for each $\psi \in \mathcal{W}$. Assume that $F^{\mathcal{V}} \tilde{\theta}_\varepsilon K^{\mathcal{V}} = T^{\mathcal{V} \cup \mathcal{W}} = T^{\mathcal{V}}$, where

$$T^{\mathcal{V}}(\lambda) = \begin{cases} (\mathfrak{S}_1', \mathfrak{S}_2') & \lambda \in \mathcal{W} - \mathcal{W} = \emptyset \\ (U_1, U_2) & \lambda \in \mathcal{W} - \mathcal{W} = \emptyset \\ (\mathfrak{S}_1' \cap U_1, \mathfrak{S}_2' \cap U_2) & \lambda \in \mathcal{W} \cap \mathcal{W} = \mathcal{W} \end{cases}$$

for each $\lambda \in \mathcal{W}$. Hence,

$$T^{\mathcal{V}}(\lambda) = (\mathfrak{S}_1' \cap \emptyset, \mathfrak{S}_2' \cap \emptyset) = (\emptyset, \emptyset) = \emptyset^{\mathcal{V}}(\lambda),$$

implying that $T^{\mathcal{V}} = \emptyset^{\mathcal{V}}$.

Proposition 3.1.18 Let $F^{\mathcal{V}}$ be a BSS over U_1, U_2 . Then, $F^{\mathcal{V}} \tilde{\theta}_\varepsilon (F^{\mathcal{V}})^r = (F^{\mathcal{V}})^r \tilde{\theta}_\varepsilon F^{\mathcal{V}} = \emptyset^{\mathcal{V}}$.

Proof: Let $F^{\mathcal{V}}(\psi) = (\mathfrak{S}_1', \mathfrak{S}_2')$ for each $\psi \in \mathcal{W}$ and $(F^{\mathcal{V}})^r = K^{\mathcal{V}}$. Then, $K^{\mathcal{V}}(\psi) = (U_1 - \mathfrak{S}_1', U_2 - \mathfrak{S}_2')$. Assume that $F^{\mathcal{V}} \tilde{\theta}_\varepsilon K^{\mathcal{V}} = T^{\mathcal{V} \cup \mathcal{W}} = T^{\mathcal{V}}$, where

$$T^{\mathcal{V}}(\lambda) = \begin{cases} (\mathfrak{S}_1', \mathfrak{S}_2') & \lambda \in \mathcal{W} - \mathcal{W} = \emptyset \\ ((U_1 - \mathfrak{S}_1)', (U_2 - \mathfrak{S}_2)') & \lambda \in \mathcal{W} - \mathcal{W} = \emptyset \\ (\mathfrak{S}_1' \cap (U_1 - \mathfrak{S}_1)', \mathfrak{S}_2' \cap (U_2 - \mathfrak{S}_2)') & \lambda \in \mathcal{W} \cap \mathcal{W} = \mathcal{W} \end{cases}$$

for each $\lambda \in \mathcal{W}$. Thereby,

$$T^{\mathcal{V}}(\lambda) = (\mathfrak{S}_1' \cap (U_1 - \mathfrak{S}_1)', \mathfrak{S}_2' \cap (U_2 - \mathfrak{S}_2)') = (\emptyset, \emptyset)$$

for each $\lambda \in \mathcal{W}$. Thus, $T^{\mathcal{V}} = \emptyset^{\mathcal{V}}$.

Proposition 3.1.19 Let $F^{\mathcal{V}}$ and $G^{\mathcal{W}}$ be BSSs over U_1, U_2 . Then, $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{W}} = \mathcal{U}^{\mathcal{W}}$ if and only if $F^{\mathcal{V}} = \emptyset^{\mathcal{V}}$ and $G^{\mathcal{W}} = \emptyset^{\mathcal{W}}$.

Proof: Let $F^{\mathcal{V}}(\psi) = (\mathfrak{S}_1', \mathfrak{S}_2')$, $G^{\mathcal{W}}(\psi) = (\mathcal{L}_1', \mathcal{L}_2')$ and $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{W}} = T^{\mathcal{V}}$, where

$$T^{\mathcal{V}}(\lambda) = \begin{cases} (\mathfrak{S}_1', \mathfrak{S}_2') & \lambda \in \mathcal{W} - \mathcal{W} = \emptyset \\ (\mathcal{L}_1', \mathcal{L}_2') & \lambda \in \mathcal{W} - \mathcal{W} = \emptyset \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in \mathcal{W} \cap \mathcal{W} = \mathcal{W} \end{cases}$$

for each $\lambda \in \mathcal{W}$. Since $T^{\mathcal{V}} = \mathcal{U}^{\mathcal{W}}$, $T^{\mathcal{V}}(\lambda) = (U_1, U_2)$ for each $\lambda \in \mathcal{W} \cup \mathcal{W}$. Thus, for each $\lambda \in \mathcal{W}$, $T^{\mathcal{V}}(\lambda) = (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') = (U_1, U_2)$. Thus, $(\mathfrak{S}_1', \mathfrak{S}_2') = (\mathcal{L}_1', \mathcal{L}_2') = (\emptyset, \emptyset)$. Thereby, $F^{\mathcal{V}} = \emptyset^{\mathcal{V}}$ and $G^{\mathcal{W}} = \emptyset^{\mathcal{W}}$.

Conversely, let $F^{\mathcal{V}} = \emptyset^{\mathcal{V}}$ and $G^{\mathcal{W}} = \emptyset^{\mathcal{W}}$. Then, $F^{\mathcal{V}}(\psi) = (\emptyset, \emptyset)$ for each $\psi \in \mathcal{W}$, and $G^{\mathcal{W}}(\psi) = (\emptyset, \emptyset)$ for all $\psi \in \mathcal{W}$. Let $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{W}} = K^{\mathcal{W}}$, where

$$K^{\mathcal{W} \cup \mathcal{W}}(\lambda) = \begin{cases} (\emptyset, \emptyset) & \lambda \in \mathcal{W} - \mathcal{W} = \emptyset \\ (\emptyset, \emptyset) & \lambda \in \mathcal{W} - \mathcal{W} = \emptyset \\ (\emptyset' \cap \emptyset', \emptyset' \cap \emptyset') & \lambda \in \mathcal{W} \cap \mathcal{W} = \mathcal{W} \end{cases}$$

for each $\lambda \in \mathcal{W}$. Thus,

$$K^{\mathcal{W}}(\lambda) = (U_1, U_2) = \mathcal{U}^{\mathcal{W}}(\lambda)$$

implying that $K^{\mathcal{W}} = \mathcal{U}^{\mathcal{W}}$.

Proposition 3.1.20 Let $F^{\mathcal{V}}$ and $G^{\mathcal{W}}$ be BSSs over U_1, U_2 . Then, $\emptyset^{\mathcal{W}} \tilde{\subseteq} F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{W}}$, $\emptyset^{\mathcal{W}} \tilde{\subseteq} F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{W}}$, and $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{W}} \tilde{\subseteq} \mathcal{U}^{\mathcal{W} \cup \mathcal{W}}$.

Proof: Since the empty set is a subset of every set, and the universal set is a superset of every set, the proof is straightforward.

Theorem 3.1.21 Let $F^{\mathcal{V}}$, $G^{\mathcal{V}}$, $H^{\mathcal{V}}$ and $K^{\mathcal{V}}$ be BSSs over U_1, U_2 . Then,

- $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{V}} \tilde{\subseteq} (F^{\mathcal{V}})^r$ and $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{V}} \tilde{\subseteq} (G^{\mathcal{V}})^r$.
- $F^{\mathcal{V}} \tilde{\subseteq} G^{\mathcal{V}}$ if and only if $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{V}} = (G^{\mathcal{V}})^r$.
- If $F^{\mathcal{V}} \tilde{\subseteq} G^{\mathcal{V}}$, then $G^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{V}} \tilde{\subseteq} F^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{V}}$.
- If $G^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}} \tilde{\subseteq} F^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}}$, then $F^{\mathcal{V}} \tilde{\subseteq} G^{\mathcal{V}}$ needs not be true.
- If $F^{\mathcal{V}} \tilde{\subseteq} G^{\mathcal{V}}$ and $H^{\mathcal{V}} \tilde{\subseteq} K^{\mathcal{V}}$, then $G^{\mathcal{V}} \tilde{\theta}_\varepsilon K^{\mathcal{V}} \tilde{\subseteq} F^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{V}}$.

Proof: (i) Let $F^{\mathcal{V}}(\psi) = (\mathfrak{S}_1', \mathfrak{S}_2')$, $G^{\mathcal{V}}(\psi) = (\mathcal{L}_1', \mathcal{L}_2')$ for each $\psi \in \mathcal{W}$, and $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{V}} = T^{\mathcal{V} \cup \mathcal{V}} = T^{\mathcal{V}}$, where

$$T^{\mathcal{V}}(\lambda) = \begin{cases} (\mathfrak{S}_1', \mathfrak{S}_2') & \lambda \in \mathcal{W} - \mathcal{W} = \emptyset \\ (\mathcal{L}_1', \mathcal{L}_2') & \lambda \in \mathcal{W} - \mathcal{W} = \emptyset \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in \mathcal{W} \cap \mathcal{W} = \mathcal{W} \end{cases}$$

Thus,

$$T^{\mathcal{V}}(\lambda) = (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') \subseteq (\mathfrak{S}_1', \mathfrak{S}_2') = (F^{\mathcal{V}})^r(\lambda)$$

for each $\lambda \in \mathcal{W}$. Hence, $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{V}} \tilde{\subseteq} (F^{\mathcal{V}})^r$. Moreover, since

$$T^{\mathcal{V}}(\lambda) = (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') \subseteq (\mathcal{L}_1', \mathcal{L}_2') = (G^{\mathcal{V}})^r(\lambda)$$

$F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{V}} \tilde{\subseteq} (G^{\mathcal{V}})^r$ is obtained.

(ii) Let $F^{\mathcal{V}}(\psi) = (\mathfrak{S}_1', \mathfrak{S}_2')$, $G^{\mathcal{V}}(\psi) = (\mathcal{L}_1', \mathcal{L}_2')$ for each $\psi \in \mathcal{W}$, and $F^{\mathcal{V}} \tilde{\subseteq} G^{\mathcal{V}}$. Then, $F^{\mathcal{V}}(\psi) \subseteq G^{\mathcal{V}}(\psi)$, for each $\psi \in \mathcal{W}$. Let $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{V}} = T^{\mathcal{V}}$, where

$$T^{\mathcal{V}}(\psi) = \begin{cases} (\mathfrak{S}_1', \mathfrak{S}_2') & \psi \in \mathcal{W} - \mathcal{W} = \emptyset \\ (\mathcal{L}_1', \mathcal{L}_2') & \psi \in \mathcal{W} - \mathcal{W} = \emptyset \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \psi \in \mathcal{W} \cap \mathcal{W} = \mathcal{W} \end{cases}$$

Thus,

$$T^{\mathcal{V}}(\psi) = (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') = (\mathcal{L}_1', \mathcal{L}_2') = (G^{\mathcal{V}})^r(\psi)$$

for each $\psi \in \mathcal{W}$, implying that $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{V}} = (G^{\mathcal{V}})^r$.

Conversely, let $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{V}} = (G^{\mathcal{V}})^r$, and $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{V}} = K^{\mathcal{V}}$. Thus,

$$K^{\mathcal{W}}(\mathcal{Y}) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \mathcal{Y} \in \mathcal{W} - \mathcal{V} = \emptyset \\ (\mathcal{L}_1, \mathcal{L}_2) & \mathcal{Y} \in \mathcal{W} - \mathcal{V} = \emptyset \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \mathcal{Y} \in \mathcal{W} \cap \mathcal{V} = \mathcal{V} \end{cases}$$

for each $\mathcal{Y} \in \mathcal{W}$. Since $K^{\mathcal{V}} = F^{\mathcal{V}}$, then $(\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') = (\mathcal{L}_1', \mathcal{L}_2')$, implying that $\mathcal{L}_1' \subseteq \mathfrak{S}_1'$ and $\mathcal{L}_2' \subseteq \mathfrak{S}_2'$. That is, $F^{\mathcal{V}}(\mathcal{Y}) \subseteq G^{\mathcal{V}}(\mathcal{Y})$ for each $\mathcal{Y} \in \mathcal{W}$. Hence, $F^{\mathcal{V}} \subseteq G^{\mathcal{V}}$.

(iii) Let $F^{\mathcal{W}}(\mathcal{Y}) = (\mathfrak{S}_1, \mathfrak{S}_2)$, $G^{\mathcal{W}}(\mathcal{Y}) = (\mathcal{L}_1, \mathcal{L}_2)$, $H^{\mathcal{W}}(\mathcal{Y}) = (\mathcal{B}_1, \mathcal{B}_2)$ for each $\mathcal{Y} \in \mathcal{W}$, and $F^{\mathcal{W}} \subseteq G^{\mathcal{W}}$. Then, $F^{\mathcal{W}}(\mathcal{Y}) \subseteq G^{\mathcal{W}}(\mathcal{Y})$ for each $\mathcal{Y} \in \mathcal{W}$. Let $F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{W}} = T^{\mathcal{W}}$, where

$$T^{\mathcal{W}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - \mathcal{V} = \emptyset \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in \mathcal{W} - \mathcal{V} = \emptyset \\ (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') & \lambda \in \mathcal{W} \cap \mathcal{V} = \mathcal{V} \end{cases}$$

Let $G^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{W}} = R^{\mathcal{W}}$, where

$$R^{\mathcal{W}}(\lambda) = \begin{cases} (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{W} - \mathcal{V} = \emptyset \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in \mathcal{W} - \mathcal{V} = \emptyset \\ (\mathcal{L}_1' \cap \mathcal{B}_1', \mathcal{L}_2' \cap \mathcal{B}_2') & \lambda \in \mathcal{W} \cap \mathcal{V} = \mathcal{V} \end{cases}$$

for each $\lambda \in \mathcal{W}$. Since $R^{\mathcal{W}}(\lambda) = ((\mathcal{L}_1' \cap \mathcal{B}_1', \mathcal{L}_2' \cap \mathcal{B}_2') \subseteq (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2')) = T^{\mathcal{W}}(\lambda)$ and $R^{\mathcal{W}}(\lambda) \subseteq T^{\mathcal{W}}(\lambda)$ for each $\lambda \in \mathcal{W}$. Thus, $G^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{W}} \subseteq F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{W}}$.

(iv) Let $U_1 = \{t_1, t_2, t_3, t_4\}$, $U_2 = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, $\mathcal{W} = \{e_1, e_3, e_5\}$ and $F^{\mathcal{W}}$, $G^{\mathcal{W}}$ and $H^{\mathcal{W}}$ be BSSs over U_1, U_2 be defined as follows:

$$\begin{aligned} F^{\mathcal{W}} &= \{(e_1, (\{t_1, t_4\}, \{h_1, h_2\})), (e_3, (\{t_1, t_2, t_3\}, \{h_3\})), \\ &\quad (e_5, (\{t_2\}, \{h_1, h_2, h_3\}))\} \\ G^{\mathcal{W}} &= \{(e_1, (\{t_1\}, \{h_1, h_3\})), (e_3, (\{t_2, t_3\}, \{h_2\})), \\ &\quad (e_5, (\{t_3\}, \{h_1, h_2\}))\} \\ H^{\mathcal{W}} &= \{(e_1, (U_1, U_2)), (e_3, (U_1, U_2)), (e_5, (U_1, U_2))\} \end{aligned}$$

Let $F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{W}} = T^{\mathcal{W}}$, where

$$T^{\mathcal{W}} = \{(e_1, (\emptyset, \emptyset)), (e_3, (\emptyset, \emptyset)), (e_5, (\emptyset, \emptyset))\}$$

for each $\lambda \in \mathcal{W} = \{e_1, e_3, e_5\}$. Let $G^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{W}} = R^{\mathcal{W}}$, where

$$R^{\mathcal{W}} = \{(e_1, (\emptyset, \emptyset)), (e_3, (\emptyset, \emptyset)), (e_5, (\emptyset, \emptyset))\}$$

for each $\lambda \in \mathcal{W} = \{e_1, e_3, e_5\}$. Here, $G^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{W}} \subseteq F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{W}}$, however $F^{\mathcal{W}} \subseteq G^{\mathcal{W}}$ is not satisfied.

(v) Let $F^{\mathcal{V}}(\mathcal{Y}) = (\mathfrak{S}_1, \mathfrak{S}_2)$, $G^{\mathcal{V}}(\mathcal{Y}) = (\mathcal{L}_1, \mathcal{L}_2)$, $H^{\mathcal{V}}(\mathcal{Y}) = (\mathcal{B}_1, \mathcal{B}_2)$, $K^{\mathcal{V}}(\mathcal{Y}) = (\mathcal{K}_1, \mathcal{K}_2)$ for each $\mathcal{Y} \in \mathcal{V}$, $F^{\mathcal{V}} \subseteq G^{\mathcal{V}}$ and $H^{\mathcal{V}} \subseteq K^{\mathcal{V}}$. Then, $F^{\mathcal{V}}(\mathcal{Y}) \subseteq G^{\mathcal{V}}(\mathcal{Y})$ and $H^{\mathcal{V}}(\mathcal{Y}) \subseteq K^{\mathcal{V}}(\mathcal{Y})$ for each $\mathcal{Y} \in \mathcal{V}$. Let $F^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{V}} = T^{\mathcal{V}}$, where

$$T^{\mathcal{V}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{V} - \mathcal{W} = \emptyset \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in \mathcal{V} - \mathcal{W} = \emptyset \\ (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') & \lambda \in \mathcal{V} \cap \mathcal{W} = \mathcal{W} \end{cases}$$

Let $G^{\mathcal{V}} \tilde{\theta}_\varepsilon K^{\mathcal{V}} = R^{\mathcal{V}}$, where

$$R^{\mathcal{V}}(\lambda) = \begin{cases} (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{V} - \mathcal{W} = \emptyset \\ (\mathcal{K}_1, \mathcal{K}_2) & \lambda \in \mathcal{V} - \mathcal{W} = \emptyset \\ (\mathcal{L}_1' \cap \mathcal{K}_1', \mathcal{L}_2' \cap \mathcal{K}_2') & \lambda \in \mathcal{V} \cap \mathcal{W} = \mathcal{W} \end{cases}$$

Since $R^{\mathcal{V}}(\lambda) = (\mathcal{L}_1' \cap \mathcal{K}_1', \mathcal{L}_2' \cap \mathcal{K}_2') \subseteq (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') = T^{\mathcal{V}}(\lambda)$ for each $\lambda \in \mathcal{V}$, $G^{\mathcal{V}} \tilde{\theta}_\varepsilon K^{\mathcal{V}} \subseteq F^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{V}}$ is obtained.

3.2 Distributions of binary extended theta operation over other types of binary soft set operations

In this subsection, the distributions of binary extended theta operation over other types of BSS operations are provided, and the algebraic structures formed by the binary extended theta operation of BSSs in $S^E(U_1, U_2)$ are thoroughly examined.

Theorem 3.2.1 Let $F^{\mathcal{W}}, G^{\mathcal{W}}$, and $H^{\mathcal{Q}}$ be BSSs over U_1, U_2 . Then, the following distributions hold, where $\mathcal{W} \cap \mathcal{W} \cap \mathcal{Q} = \mathcal{W} \cap (\mathcal{W} \cap \mathcal{Q}) = \emptyset$;

- 1) $F^{\mathcal{W}} \tilde{\theta}_\varepsilon (G^{\mathcal{W}} \tilde{\cup}_\varepsilon H^{\mathcal{Q}}) = (F^{\mathcal{W}} \tilde{\theta}_\varepsilon G^{\mathcal{W}}) \tilde{\cup}_\varepsilon (F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}})$
- 2) $F^{\mathcal{W}} \tilde{\theta}_\varepsilon (G^{\mathcal{W}} \tilde{\cap}_\varepsilon H^{\mathcal{Q}}) = (F^{\mathcal{W}} \tilde{\theta}_\varepsilon G^{\mathcal{W}}) \tilde{\cap}_\varepsilon (F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}})$

Proof: (1) Let $F^{\mathcal{W}}(\mathcal{Y}) = (\mathfrak{S}_1, \mathfrak{S}_2)$ for each $\mathcal{Y} \in \mathcal{W}$, $G^{\mathcal{W}}(\mathcal{W}) = (\mathcal{L}_1, \mathcal{L}_2)$ for each $\mathcal{W} \in \mathcal{W}$, $H^{\mathcal{Q}}(\mathcal{Q}) = (\mathcal{B}_1, \mathcal{B}_2)$ for each $\mathcal{Q} \in \mathcal{Q}$. Let $G^{\mathcal{W}} \tilde{\cap}_\varepsilon H^{\mathcal{Q}} = T^{\mathcal{W} \cup \mathcal{Q}}$, where

$$T^{\mathcal{W} \cup \mathcal{Q}}(\lambda) = \begin{cases} (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{W} - \mathcal{Q} \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in \mathcal{Q} - \mathcal{W} \\ (\mathcal{L}_1 \cup \mathcal{B}_1, \mathcal{L}_2 \cup \mathcal{B}_2) & \lambda \in \mathcal{W} \cap \mathcal{Q} \end{cases}$$

for each $\lambda \in \mathcal{W} \cup \mathcal{Q}$ and $F^{\mathcal{W}} \tilde{\theta}_\varepsilon T^{\mathcal{W} \cup \mathcal{Q}} = R^{\mathcal{W} \cup (\mathcal{W} \cup \mathcal{Q})}$, where

$$R^{\mathcal{W} \cup (\mathcal{W} \cup \mathcal{Q})}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - (\mathcal{W} \cup \mathcal{Q}) = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in (\mathcal{W} - \mathcal{Q}) - \mathcal{W} = \mathcal{W}' \cap \mathcal{W} \cap \mathcal{Q}' \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in (\mathcal{Q} - \mathcal{W}) - \mathcal{W} = \mathcal{W}' \cap \mathcal{W}' \cap \mathcal{Q} \\ (\mathcal{L}_1 \cup \mathcal{B}_1, \mathcal{L}_2 \cup \mathcal{B}_2) & \lambda \in (\mathcal{W} \cap \mathcal{Q}) - \mathcal{W} = \mathcal{W}' \cap \mathcal{W} \cap \mathcal{Q} \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in \mathcal{W} \cap (\mathcal{W} - \mathcal{Q}) = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') & \lambda \in \mathcal{W} \cap (\mathcal{Q} - \mathcal{W}) = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q} \\ (\mathfrak{S}_1' \cap (\mathcal{L}_1' \cap \mathcal{B}_1'), \mathfrak{S}_2' \cap (\mathcal{L}_2' \cap \mathcal{B}_2')) & \lambda \in \mathcal{W} \cap (\mathcal{W} \cap \mathcal{Q}) = \mathcal{W} \cap \mathcal{W} \cap \mathcal{Q} \end{cases}$$

for each $\lambda \in \mathcal{W} \cup (\mathcal{W} \cup \mathcal{Q})$. Assume that $F^{\mathcal{W}} \tilde{\theta}_\varepsilon G^{\mathcal{W}} = K^{\mathcal{W} \cup \mathcal{W}}$, where

$$K^{\mathcal{W} \cup \mathcal{W}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - \mathcal{W} \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{W} - \mathcal{W} \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in \mathcal{W} \cap \mathcal{W} \end{cases}$$

for all $\lambda \in \mathcal{W} \cup \mathcal{Q}$ and $F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}} = S^{\mathcal{W} \cup \mathcal{Q}}$, where

$$S^{\mathcal{W} \cup \mathcal{Q}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - \mathcal{Q} \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in \mathcal{Q} - \mathcal{W} \\ (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') & \lambda \in \mathcal{W} \cap \mathcal{Q} \end{cases}$$

for each $\lambda \in \mathcal{W} \cup \mathcal{Q}$. Let $K^{\mathcal{W} \cup \mathcal{W}} \tilde{\theta}_\varepsilon S^{\mathcal{W} \cup \mathcal{Q}} = W^{(\mathcal{W} \cup \mathcal{W}) \cup (\mathcal{W} \cup \mathcal{Q})}$, where for each $\lambda \in (\mathcal{W} \cup \mathcal{W}) \cup (\mathcal{W} \cup \mathcal{Q})$,

$$W^{(\mathcal{W} \cup \mathcal{W}) \cup (\mathcal{W} \cup \mathcal{Q})}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (\mathcal{W} - \mathcal{W}) - (\mathcal{W} \cup \mathcal{Q}) \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in (\mathcal{W} - \mathcal{W}) - (\mathcal{W} \cup \mathcal{Q}) \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in (\mathcal{W} \cap \mathcal{W}) - (\mathcal{W} \cup \mathcal{Q}) \\ (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (\mathcal{W} - \mathcal{Q}) - (\mathcal{W} \cup \mathcal{W}) \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in (\mathcal{Q} - \mathcal{W}) - (\mathcal{W} \cup \mathcal{W}) \\ (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') & \lambda \in (\mathcal{W} \cap \mathcal{Q}) - (\mathcal{W} \cup \mathcal{W}) \\ (\mathfrak{S}_1 \cup \mathfrak{S}_1', \mathfrak{S}_2 \cup \mathfrak{S}_2') & \lambda \in (\mathcal{W} - \mathcal{W}) \cap (\mathcal{W} - \mathcal{Q}) \\ (\mathfrak{S}_1 \cup \mathcal{B}_1, \mathfrak{S}_2 \cup \mathcal{B}_2) & \lambda \in (\mathcal{W} - \mathcal{W}) \cap (\mathcal{Q} - \mathcal{W}) \\ (\mathfrak{S}_1 \cup (\mathfrak{S}_1' \cap \mathcal{B}_1'), \mathfrak{S}_2 \cup (\mathfrak{S}_2' \cap \mathcal{B}_2')) & \lambda \in (\mathcal{W} - \mathcal{W}) \cap (\mathcal{W} \cap \mathcal{Q}) \\ (\mathcal{L}_1 \cup \mathfrak{S}_1, \mathcal{L}_2 \cup \mathfrak{S}_2) & \lambda \in (\mathcal{W} - \mathcal{W}) \cap (\mathcal{W} - \mathcal{Q}) \\ (\mathcal{L}_1 \cup \mathcal{B}_1, \mathcal{L}_2 \cup \mathcal{B}_2) & \lambda \in (\mathcal{W} - \mathcal{W}) \cap (\mathcal{Q} - \mathcal{W}) \\ (\mathcal{L}_1 \cup (\mathfrak{S}_1' \cap \mathcal{B}_1'), \mathcal{L}_2 \cup (\mathfrak{S}_2' \cap \mathcal{B}_2')) & \lambda \in (\mathcal{W} - \mathcal{W}) \cap (\mathcal{W} \cap \mathcal{Q}) \\ ((\mathfrak{S}_1' \cap \mathcal{L}_1') \cup \mathfrak{S}_1, (\mathfrak{S}_2' \cap \mathcal{L}_2') \cup \mathfrak{S}_2) & \lambda \in (\mathcal{W} \cap \mathcal{W}) \cap (\mathcal{W} - \mathcal{Q}) \\ ((\mathfrak{S}_1' \cap \mathcal{L}_1') \cup \mathcal{B}_1, (\mathfrak{S}_2' \cap \mathcal{L}_2') \cup \mathcal{B}_2) & \lambda \in (\mathcal{W} \cap \mathcal{W}) \cap (\mathcal{Q} - \mathcal{W}) \\ ((\mathfrak{S}_1' \cap \mathcal{L}_1') \cup (\mathfrak{S}_1' \cap \mathcal{B}_1'), (\mathfrak{S}_2' \cap \mathcal{L}_2') \cup (\mathfrak{S}_2' \cap \mathcal{B}_2')) & \lambda \in (\mathcal{W} \cap \mathcal{W}) \cap (\mathcal{W} \cap \mathcal{Q}) \end{cases}$$

Thus,

$$W^{(\mathcal{W} \cup \mathcal{W}) \cap (\mathcal{W} \cup \mathcal{Q})}(\lambda) = \begin{cases} (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in (\mathcal{W} - \mathcal{W}) \cap (\mathcal{W} \cup \mathcal{Q}) = \mathcal{W}' \cap \mathcal{W} \cap \mathcal{Q}' \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in (\mathcal{Q} - \mathcal{W}) - (\mathcal{W} \cup \mathcal{W}) = \mathcal{W}' \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (\mathcal{W} - \mathcal{W}) \cap (\mathcal{W} - \mathcal{Q}) = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathfrak{S}_1 \cup \mathcal{B}_1', \mathfrak{S}_2 \cup \mathcal{B}_2') & \lambda \in (\mathcal{W} - \mathcal{W}) \cap (\mathcal{W} \cap \mathcal{Q}) = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathcal{L}_1 \cup \mathcal{B}_1, \mathcal{L}_2 \cup \mathcal{B}_2) & \lambda \in (\mathcal{W} - \mathcal{W}) \cap (\mathcal{Q} - \mathcal{W}) = \mathcal{W}' \cap \mathcal{W} \cap \mathcal{Q}' \\ (\mathcal{L}_1 \cup \mathfrak{S}_1, \mathcal{L}_2 \cup \mathfrak{S}_2) & \lambda \in (\mathcal{W} \cap \mathcal{W}) \cap (\mathcal{W} - \mathcal{Q}) = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathfrak{S}_1' \cap (\mathcal{L}_1' \cup \mathcal{B}_1'), \mathfrak{S}_2' \cap (\mathcal{L}_2' \cup \mathcal{B}_2')) & \lambda \in (\mathcal{W} \cap \mathcal{W}) \cap (\mathcal{W} \cap \mathcal{Q}) = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' \end{cases}$$

Here R and W are piecewise functions that form both sides of the equality. When examining the parameters line by line, the equality can be achieved by selecting the corresponding parameter sets as empty in the mismatched lines. This situation indicates that certain parameter separation conditions are necessary for the distribution laws to be valid.

Thereby, $R^{\mathcal{W} \cup (\mathcal{W} \cup \mathcal{Q})} = W^{(\mathcal{W} \cup \mathcal{W}) \cap (\mathcal{W} \cup \mathcal{Q})}$, where $\mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q} = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}' = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q} = \emptyset$. It is obvious that $\mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q} = \mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q}'$ means that $\mathcal{W} \cap (\mathcal{W} \Delta \mathcal{Q}) = \emptyset$.

Example 3.2.2 Let $U_1 = \{t_1, t_2, t_3, t_4, t_5\}$, $U_2 = \{h_1, h_2, h_3, h_4\}$, $E = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$, $\mathcal{W} = \{\psi_1\}$, $\mathcal{W}' = \{\psi_2, \psi_3\}$ and $\mathcal{Q} = \{\psi_2, \psi_4\}$ the BSSs $F^{\mathcal{W}}$, $G^{\mathcal{W}'}$ and $H^{\mathcal{Q}}$ over U_1, U_2 be defined as follows:

$$F^{\mathcal{W}} = \{(\psi_1, (\{t_1, t_2, t_3\}, \{h_1, h_2\}))\}$$

$$G^{\mathcal{W}'} = \{(\psi_2, (U_1, \{h_2, h_3, h_4\})), (\psi_3, (\{t_3, t_4\}, \{h_2, h_3, h_4\}))\}$$

$$H^{\mathcal{Q}} = \{(\psi_2, (\{t_1, t_5\}, U_2)), (\psi_4, (\emptyset, \emptyset))\}$$

One can easily show that

$$F^{\mathcal{W}} \tilde{\theta}_\varepsilon (G^{\mathcal{W}'} \tilde{\theta}_\varepsilon H^{\mathcal{Q}}) = (F^{\mathcal{W}} \tilde{\theta}_\varepsilon G^{\mathcal{W}'}) \tilde{\theta}_\varepsilon (F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}}) = \{(\psi_1, (\{t_1, t_2, t_3\}, \{h_1, h_2\})), (\psi_2, (U_1, U_2)), (\psi_3, (\{t_3, t_4\}, \{h_2, h_3, h_4\})), (\psi_4, (\emptyset, \emptyset))\}$$

Here note that $\mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q} = \mathcal{W} \cap (\mathcal{W} \Delta \mathcal{Q}) = \emptyset$. Thus, the distribution holds. However, we have the following example:

Example 3.2.3 Let $U_1 = \{t_1, t_2, t_3, t_4, t_5\}$, $U_2 = \{h_1, h_2, h_3, h_4\}$, $E = \{\psi_1, \psi_2, \psi_3\}$, $\mathcal{W} = \{\psi_1, \psi_2\}$, $\mathcal{W}' = \{\psi_1, \psi_3\}$ and $\mathcal{Q} = \{\psi_1, \psi_2, \psi_3\}$ the BSSs $F^{\mathcal{W}}$, $G^{\mathcal{W}'}$ and $H^{\mathcal{Q}}$ over U_1, U_2 be defined as follows:

$$F^{\mathcal{W}} = \{(\psi_1, (\{t_1, t_2\}, \{h_2\})), (\psi_2, (\emptyset, \{h_2, h_3\}))\}$$

$$G^{\mathcal{W}'} = \{(\psi_1, (\emptyset, \{h_2, h_4\})), (\psi_3, (\{t_3\}, \emptyset))\}$$

$$H^{\mathcal{Q}} = \{(\psi_1, (\{t_1, t_5\}, \{h_2, h_3\})), (\psi_2, (\{t_1, t_5\}, \{h_2, h_3\})), (\psi_3, (\emptyset, \emptyset))\}$$

One can easily show that

$$F^{\mathcal{W}} \tilde{\theta}_\varepsilon (G^{\mathcal{W}'} \tilde{\theta}_\varepsilon H^{\mathcal{Q}}) = \{(\psi_1, (\{t_3, t_4\}, \{h_1\})), (\psi_2, (\{t_2, t_3, t_4\}, \{h_1, h_4\})), (\psi_3, (\{t_3\}, \emptyset))\}$$

and

$$(F^{\mathcal{W}} \tilde{\theta}_\varepsilon G^{\mathcal{W}'}) \tilde{\theta}_\varepsilon (F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}}) = \{(\psi_1, (\{t_3, t_4, t_5\}, \{h_1, h_3, h_4\})), (\psi_2, (\{t_2, t_3, t_4\}, U_2)), (\psi_3, (\{t_3\}, \emptyset))\}$$

It is observed that

$$F^{\mathcal{W}} \tilde{\theta}_\varepsilon (G^{\mathcal{W}'} \tilde{\theta}_\varepsilon H^{\mathcal{Q}}) \neq (F^{\mathcal{W}} \tilde{\theta}_\varepsilon G^{\mathcal{W}'}) \tilde{\theta}_\varepsilon (F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}})$$

Here note that $\mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q} = \{1\} \neq \emptyset$ and $\mathcal{W} \cap (\mathcal{W} \Delta \mathcal{Q}) = \{2\} \neq \emptyset$. Thus, the distribution does not hold.

Theorem 3.2.4 Let $F^{\mathcal{W}}$, $G^{\mathcal{W}'}$, and $H^{\mathcal{Q}}$ be BSSs over U_1, U_2 . Then, the following distributions hold, where $\mathcal{W} \cap \mathcal{W}' \cap \mathcal{Q} = (\mathcal{W} \Delta \mathcal{W}) \cap \mathcal{Q} = \emptyset$;

$$1) (F^{\mathcal{W}} \tilde{\theta}_\varepsilon G^{\mathcal{W}'}) \tilde{\theta}_\varepsilon H^{\mathcal{Q}} = (F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}}) \tilde{\theta}_\varepsilon (G^{\mathcal{W}'} \tilde{\theta}_\varepsilon H^{\mathcal{Q}})$$

$$2) (F^{\mathcal{W}} \tilde{\theta}_\varepsilon G^{\mathcal{W}'}) \tilde{\theta}_\varepsilon H^{\mathcal{Q}} = (F^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}}) \tilde{\theta}_\varepsilon (G^{\mathcal{W}'} \tilde{\theta}_\varepsilon H^{\mathcal{Q}})$$

Proof: (1) Let $F^{\mathcal{W}}(\psi) = (\mathfrak{S}_1, \mathfrak{S}_2)$ for each $\psi \in \mathcal{W}$, $G^{\mathcal{W}'}(\psi) = (\mathcal{L}_1, \mathcal{L}_2)$ for each $\psi \in \mathcal{W}'$, $H^{\mathcal{Q}}(\psi) = (\mathcal{B}_1, \mathcal{B}_2)$ for each $\psi \in \mathcal{Q}$ and $F^{\mathcal{W}} \tilde{\theta}_\varepsilon G^{\mathcal{W}'} = T^{\mathcal{W} \cup \mathcal{W}'}$, where $\lambda \in \mathcal{W} \cup \mathcal{W}'$,

$$T^{\mathcal{W} \cup \mathcal{W}'}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{W} - \mathcal{W}' \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{W}' - \mathcal{W} \\ (\mathfrak{S}_1 \cap \mathcal{L}_1, \mathfrak{S}_2 \cap \mathcal{L}_2) & \lambda \in \mathcal{W} \cap \mathcal{W}' \end{cases}$$

Let $T^{\mathcal{W} \cup \mathcal{W}'} \tilde{\theta}_\varepsilon H^{\mathcal{Q}} = R^{(\mathcal{W} \cup \mathcal{W}') \cup \mathcal{Q}}$, where $\lambda \in (\mathcal{W} \cup \mathcal{W}') \cup \mathcal{Q}$,

$$R^{(\mathcal{V} \cup \mathcal{W}) \cup \mathcal{Q}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (\mathcal{V} - \mathcal{W}) - \mathcal{Q} = \mathcal{V} \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in (\mathcal{W} - \mathcal{V}) - \mathcal{Q} = \mathcal{W}' \cap \mathcal{V} \cap \mathcal{Q}' \\ (\mathfrak{S}_1 \cap \mathcal{L}_1, \mathfrak{S}_2 \cap \mathcal{L}_2) & \lambda \in (\mathcal{V} \cap \mathcal{W}) - \mathcal{Q} = \mathcal{V} \cap \mathcal{W} \cap \mathcal{Q}' \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in \mathcal{Q} - (\mathcal{V} \cup \mathcal{W}) = \mathcal{V}' \cap \mathcal{W}' \cap \mathcal{Q} \\ (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') & \lambda \in (\mathcal{V} - \mathcal{W}) \cap \mathcal{Q} = \mathcal{V} \cap \mathcal{W}' \cap \mathcal{Q} \\ (\mathcal{L}_1' \cap \mathcal{B}_1', \mathcal{L}_2' \cap \mathcal{B}_2') & \lambda \in (\mathcal{W} - \mathcal{V}) \cap \mathcal{Q} = \mathcal{W}' \cap \mathcal{V} \cap \mathcal{Q}' \\ ((\mathfrak{S}_1' \cup \mathcal{L}_1') \cap \mathcal{B}_1', (\mathfrak{S}_2' \cup \mathcal{L}_2') \cap \mathcal{B}_2') & \lambda \in (\mathcal{V} \cap \mathcal{W}) \cap \mathcal{Q} = \mathcal{V} \cap \mathcal{W} \cap \mathcal{Q}' \end{cases}$$

Let's consider $(F^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}}) \tilde{\mathfrak{N}}_\varepsilon (G^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}})$. Let $F^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}} = K^{\mathcal{V} \cup \mathcal{Q}}$, where $\lambda \in \mathcal{V} \cup \mathcal{Q}$,

$$K^{\mathcal{V} \cup \mathcal{Q}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{V} - \mathcal{Q} \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in \mathcal{Q} - \mathcal{V} \\ (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') & \lambda \in \mathcal{V} \cap \mathcal{Q} \end{cases}$$

Assume that $G^{\mathcal{W}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}} = S^{\mathcal{W} \cup \mathcal{Q}}$, where $\lambda \in \mathcal{W} \cup \mathcal{Q}$,

$$S^{\mathcal{W} \cup \mathcal{Q}}(\lambda) = \begin{cases} (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{W} - \mathcal{Q} \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in \mathcal{Q} - \mathcal{W} \\ (\mathcal{L}_1' \cap \mathcal{B}_1', \mathcal{L}_2' \cap \mathcal{B}_2') & \lambda \in \mathcal{W} \cap \mathcal{Q} \end{cases}$$

Let $K^{\mathcal{V} \cup \mathcal{Q}} \tilde{\mathfrak{N}}_\varepsilon S^{\mathcal{W} \cup \mathcal{Q}} = W^{(\mathcal{V} \cup \mathcal{Q}) \cup (\mathcal{W} \cup \mathcal{Q})}$, where for each $\lambda \in (\mathcal{V} \cup \mathcal{Q}) \cup (\mathcal{W} \cup \mathcal{Q})$,

$$W^{(\mathcal{V} \cup \mathcal{Q}) \cup (\mathcal{W} \cup \mathcal{Q})}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (\mathcal{V} - \mathcal{Q}) - (\mathcal{W} \cup \mathcal{Q}) \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in (\mathcal{Q} - \mathcal{V}) - (\mathcal{W} \cup \mathcal{Q}) \\ (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') & \lambda \in (\mathcal{V} \cap \mathcal{Q}) - (\mathcal{W} \cup \mathcal{Q}) \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in (\mathcal{W} - \mathcal{Q}) - (\mathcal{V} \cup \mathcal{Q}) \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in (\mathcal{Q} - \mathcal{W}) - (\mathcal{V} \cup \mathcal{Q}) \\ (\mathcal{L}_1' \cap \mathcal{B}_1', \mathcal{L}_2' \cap \mathcal{B}_2') & \lambda \in (\mathcal{W} \cap \mathcal{Q}) - (\mathcal{V} \cup \mathcal{Q}) \\ (\mathfrak{S}_1 \cap \mathcal{L}_1, \mathfrak{S}_2 \cap \mathcal{L}_2) & \lambda \in (\mathcal{V} - \mathcal{Q}) \cap (\mathcal{W} - \mathcal{Q}) \\ (\mathfrak{S}_1 \cap \mathcal{B}_1, \mathfrak{S}_2 \cap \mathcal{B}_2) & \lambda \in (\mathcal{V} - \mathcal{Q}) \cap (\mathcal{Q} - \mathcal{W}) \\ (\mathfrak{S}_1 \cap (\mathcal{L}_1' \cap \mathcal{B}_1'), \mathfrak{S}_2 \cap (\mathcal{L}_2' \cap \mathcal{B}_2')) & \lambda \in (\mathcal{V} - \mathcal{Q}) \cap (\mathcal{W} \cap \mathcal{Q}) \\ (\mathcal{B}_1 \cap \mathcal{L}_1, \mathcal{B}_2 \cap \mathcal{L}_2) & \lambda \in (\mathcal{Q} - \mathcal{V}) \cap (\mathcal{W} - \mathcal{Q}) \\ (\mathcal{B}_1 \cap \mathcal{B}_1, \mathcal{B}_2 \cap \mathcal{B}_2) & \lambda \in (\mathcal{Q} - \mathcal{V}) \cap (\mathcal{Q} - \mathcal{W}) \\ (\mathcal{B}_1 \cap (\mathcal{L}_1' \cap \mathcal{B}_1'), \mathcal{B}_2 \cap (\mathcal{L}_2' \cap \mathcal{B}_2')) & \lambda \in (\mathcal{Q} - \mathcal{V}) \cap (\mathcal{W} \cap \mathcal{Q}) \\ ((\mathfrak{S}_1' \cap \mathcal{B}_1') \cap \mathcal{L}_1, (\mathfrak{S}_2' \cap \mathcal{B}_2') \cap \mathcal{L}_2) & \lambda \in (\mathcal{V} \cap \mathcal{Q}) \cap (\mathcal{W} - \mathcal{Q}) \\ ((\mathfrak{S}_1' \cap \mathcal{B}_1') \cap \mathcal{B}_1, (\mathfrak{S}_2' \cap \mathcal{B}_2') \cap \mathcal{B}_2) & \lambda \in (\mathcal{V} \cap \mathcal{Q}) \cap (\mathcal{Q} - \mathcal{W}) \\ ((\mathfrak{S}_1' \cap \mathcal{B}_1') \cap (\mathcal{L}_1' \cap \mathcal{B}_1'), (\mathfrak{S}_2' \cap \mathcal{B}_2') \cap (\mathcal{L}_2' \cap \mathcal{B}_2')) & \lambda \in (\mathcal{V} \cap \mathcal{Q}) \cap (\mathcal{W} \cap \mathcal{Q}) \end{cases}$$

Thus,

$$W^{(\mathcal{V} \cup \mathcal{Q}) \cup (\mathcal{W} \cup \mathcal{Q})}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (\mathcal{V} - \mathcal{Q}) - (\mathcal{W} \cup \mathcal{Q}) = \mathcal{V} \cap \mathcal{W}' \cap \mathcal{Q}' \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in (\mathcal{W} - \mathcal{Q}) - (\mathcal{V} \cup \mathcal{Q}) = \mathcal{W}' \cap \mathcal{V} \cap \mathcal{Q}' \\ (\mathfrak{S}_1 \cap \mathcal{L}_1, \mathfrak{S}_2 \cap \mathcal{L}_2) & \lambda \in (\mathcal{V} - \mathcal{Q}) \cap (\mathcal{W} - \mathcal{Q}) = \mathcal{V} \cap \mathcal{W} \cap \mathcal{Q}' \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in (\mathcal{Q} - \mathcal{V}) \cap (\mathcal{Q} - \mathcal{W}) = \mathcal{V}' \cap \mathcal{W}' \cap \mathcal{Q} \\ (\emptyset, \emptyset) & \lambda \in (\mathcal{Q} - \mathcal{V}) \cap (\mathcal{W} \cap \mathcal{Q}) = \mathcal{V}' \cap \mathcal{W} \cap \mathcal{Q}' \\ (\emptyset, \emptyset) & \lambda \in (\mathcal{V} \cap \mathcal{Q}) \cap (\mathcal{Q} - \mathcal{W}) = \mathcal{V} \cap \mathcal{W}' \cap \mathcal{Q} \\ ((\mathfrak{S}_1' \cap \mathcal{B}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{B}_2' \cap \mathcal{L}_2')) & \lambda \in (\mathcal{V} \cap \mathcal{Q}) \cap (\mathcal{W} \cap \mathcal{Q}) = \mathcal{V} \cap \mathcal{W} \cap \mathcal{Q}' \end{cases}$$

Hence, $R^{(\mathcal{V} \cup \mathcal{Q}) \cup (\mathcal{W} \cup \mathcal{Q})} = W^{(\mathcal{V} \cup \mathcal{Q}) \cup (\mathcal{W} \cup \mathcal{Q})}$, where $\mathcal{V}' \cap \mathcal{W} \cap \mathcal{Q} = \mathcal{V} \cap \mathcal{W}' \cap \mathcal{Q} = \mathcal{V} \cap \mathcal{W} \cap \mathcal{Q}' = \emptyset$. It is obvious that $\mathcal{V}' \cap \mathcal{W} \cap \mathcal{Q} = \mathcal{V} \cap \mathcal{W}' \cap \mathcal{Q} = \emptyset$ means that $(\mathcal{W} \Delta \mathcal{V}) \cap \mathcal{Q} = \emptyset$.

Theorem 3.2.5 $(S^E(U_1, U_2), \tilde{\mathfrak{U}}_\varepsilon, \tilde{\theta}_\varepsilon)$ and $(S^E(U_1, U_2), \tilde{\mathfrak{N}}_\varepsilon, \tilde{\theta}_\varepsilon)$ are multiplicatively commutative and additively idempotent semirings with the identity \emptyset^0 and without zero UCCs.

Proof. Soylu (2026) showed that $(S^E(U_1, U_2), \tilde{\mathfrak{U}}_\varepsilon)$ and $(S^E(U_1, U_2), \tilde{\mathfrak{N}}_\varepsilon)$ are commutative, idempotent monoids with the identity \emptyset_0 , that is, a bounded semilattice (hence a semigroup). $(S^E(U_1, U_2), \tilde{\theta}_\varepsilon)$ is a commutative monoid (hence a semigroup) whose identity is \emptyset_0 under the condition $\mathcal{V} \cap \mathcal{W} \cap \mathcal{Q} = \emptyset$, where $F^{\mathcal{V}}, G^{\mathcal{W}}$, and $H^{\mathcal{Q}}$ are BSSs over U_1, U_2 . Moreover, $\tilde{\theta}_\varepsilon$ distributes over $\tilde{\mathfrak{U}}_\varepsilon$ and $\tilde{\mathfrak{N}}_\varepsilon$ under the conditions $\mathcal{V} \cap \mathcal{W} \cap \mathcal{Q} = (\mathcal{V} \Delta \mathcal{W}) \cap \mathcal{Q} = \mathcal{V} \cap (\mathcal{W} \Delta \mathcal{Q}) = \emptyset$. Consequently, UCCs, $(S^E(U_1, U_2), \tilde{\mathfrak{U}}_\varepsilon, \tilde{\theta}_\varepsilon)$ and $(S^E(U_1, U_2), \tilde{\mathfrak{N}}_\varepsilon, \tilde{\theta}_\varepsilon)$ are additive idempotent multiplicatively commutative semiring without zero, but with unity.

Theorem 3.2.6 Let $F^{\mathcal{V}}, G^{\mathcal{W}}$, and $H^{\mathcal{Q}}$ be BSSs over U_1, U_2 . Then, the following distributions hold, where $\mathcal{V} \cap \mathcal{W} \cap \mathcal{Q} = \mathcal{V} \cap (\mathcal{W} \Delta \mathcal{Q}) = \emptyset$;

- 1) $F^{\mathcal{V}} \tilde{\theta}_\varepsilon (G^{\mathcal{W}} \tilde{\mathfrak{N}} H^{\mathcal{Q}}) = (F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{W}}) \tilde{\mathfrak{N}} (F^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}})$
- 2) $F^{\mathcal{V}} \tilde{\theta}_\varepsilon (G^{\mathcal{W}} \tilde{\mathfrak{U}} H^{\mathcal{Q}}) = (F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{W}}) \tilde{\mathfrak{U}} (F^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}})$

Proof: (1) Let $F^{\mathcal{V}}(\mathcal{V}) = (\mathfrak{S}_1, \mathfrak{S}_2)$ for each $\mathcal{V} \in \mathcal{V}$, $G^{\mathcal{W}}(\mathcal{W}) = (\mathcal{L}_1, \mathcal{L}_2)$ for each $\mathcal{W} \in \mathcal{W}$, $H^{\mathcal{Q}}(\mathcal{Q}) = (\mathcal{B}_1, \mathcal{B}_2)$ for each $\mathcal{Q} \in \mathcal{Q}$. Let $G^{\mathcal{W}} \tilde{\mathfrak{N}} H^{\mathcal{Q}} = T^{\mathcal{W}}$, where

$$T^{\mathcal{W}}(\lambda) = \begin{cases} (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{W} - \mathcal{Q} \\ (\mathcal{L}_1 \cap \mathcal{B}_1, \mathcal{L}_2 \cap \mathcal{B}_2) & \lambda \in \mathcal{W} \cap \mathcal{Q} \end{cases}$$

for each $\lambda \in \mathcal{W}$, and $F^{\mathcal{V}} \tilde{\theta}_\varepsilon T^{\mathcal{W}} = R^{\mathcal{V} \cup \mathcal{W}}$, where

$$R^{\mathcal{V} \cup \mathcal{W}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{V} - \mathcal{W} \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in (\mathcal{W} - \mathcal{Q}) - \mathcal{V} = \mathcal{W}' \cap \mathcal{V} \cap \mathcal{Q}' \\ (\mathcal{L}_1 \cap \mathcal{B}_1, \mathcal{L}_2 \cap \mathcal{B}_2) & \lambda \in (\mathcal{W} \cap \mathcal{Q}) - \mathcal{V} = \mathcal{W}' \cap \mathcal{W} \cap \mathcal{Q}' \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in \mathcal{V} \cap (\mathcal{W} - \mathcal{Q}) = \mathcal{V} \cap \mathcal{W}' \cap \mathcal{Q}' \\ ((\mathfrak{S}_1' \cap (\mathcal{L}_1' \cap \mathcal{B}_1'), \mathfrak{S}_2' \cap (\mathcal{L}_2' \cap \mathcal{B}_2'))) & \lambda \in \mathcal{V} \cap (\mathcal{W} \cap \mathcal{Q}) = \mathcal{V} \cap \mathcal{W} \cap \mathcal{Q}' \end{cases}$$

for each $\lambda \in \mathcal{V} \cup \mathcal{W}$. Assume that $F^{\mathcal{V}} \tilde{\theta}_\varepsilon G^{\mathcal{W}} = K^{\mathcal{V} \cup \mathcal{W}}$, where

$$K^{\mathcal{V} \cup \mathcal{W}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{V} - \mathcal{W} \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in \mathcal{W} - \mathcal{V} \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in \mathcal{V} \cap \mathcal{W} \end{cases}$$

for all $\lambda \in \mathcal{V} \cup \mathcal{W}$ and $F^{\mathcal{V}} \tilde{\theta}_\varepsilon H^{\mathcal{Q}} = S^{\mathcal{V} \cup \mathcal{Q}}$, where

$$S^{\mathcal{V} \cup \mathcal{Q}}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in \mathcal{V} - \mathcal{Q} \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in \mathcal{Q} - \mathcal{V} \\ (\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2') & \lambda \in \mathcal{V} \cap \mathcal{Q} \end{cases}$$

for each $\lambda \in \mathcal{V} \cup \mathcal{Q}$. Let $K^{\mathcal{V} \cup \mathcal{W}} \tilde{\mathfrak{N}} S^{\mathcal{V} \cup \mathcal{Q}} = W^{\mathcal{V} \cup \mathcal{W}}$, where for each $\lambda \in \mathcal{V} \cup \mathcal{W}$,

$$W^{W \cup Q}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (W - Q) - (W \cup Q) \\ (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in (W - Q) - (W \cup Q) \\ (\mathfrak{S}_1' \cap \mathcal{L}_1', \mathfrak{S}_2' \cap \mathcal{L}_2') & \lambda \in (W \cap Q) - (W \cup Q) \\ (\mathfrak{S}_1 \cap \mathfrak{S}_1', \mathfrak{S}_2 \cap \mathfrak{S}_2') & \lambda \in (W - Q) \cap (W - Q) \\ (\mathfrak{S}_1 \cap \mathcal{B}_1, \mathfrak{S}_2 \cap \mathcal{B}_2) & \lambda \in (W - Q) \cap (Q - W) \\ (\mathfrak{S}_1 \cap (\mathfrak{S}_1' \cap \mathcal{B}_1'), \mathfrak{S}_2 \cap (\mathfrak{S}_2' \cap \mathcal{B}_2')) & \lambda \in (W - Q) \cap (W \cap Q) \\ (\mathcal{L}_1 \cap \mathfrak{S}_1, \mathcal{L}_2 \cap \mathfrak{S}_2) & \lambda \in (W - Q) \cap (W - Q) \\ (\mathcal{L}_1 \cap \mathcal{B}_1, \mathcal{L}_2 \cap \mathcal{B}_2) & \lambda \in (W - Q) \cap (Q - W) \\ (\mathcal{L}_1 \cap (\mathfrak{S}_1' \cap \mathcal{B}_1'), \mathcal{L}_2 \cap (\mathfrak{S}_2' \cap \mathcal{B}_2')) & \lambda \in (W - Q) \cap (W \cap Q) \\ ((\mathfrak{S}_1' \cap \mathcal{L}_1') \cap \mathfrak{S}_1, (\mathfrak{S}_2' \cap \mathcal{L}_2') \cap \mathfrak{S}_2) & \lambda \in (W \cap Q) \cap (W - Q) \\ ((\mathfrak{S}_1' \cap \mathcal{L}_1') \cap \mathcal{B}_1, (\mathfrak{S}_2' \cap \mathcal{L}_2') \cap \mathcal{B}_2) & \lambda \in (W \cap Q) \cap (Q - W) \\ ((\mathfrak{S}_1' \cap \mathcal{L}_1') \cap (\mathcal{L}_1' \cap \mathcal{B}_1'), (\mathfrak{S}_2' \cap \mathcal{L}_2') \cap (\mathcal{L}_2' \cap \mathcal{B}_2')) & \lambda \in (W \cap Q) \cap (W \cap Q) \end{cases}$$

Thus,

$$W^{W \cup Q}(\lambda) = \begin{cases} (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in (W - Q) \cap (W \cup Q) = W' \cap W \cap Q' \\ (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (W - Q) \cap (W - Q) = W \cap W' \cap Q' \\ (\emptyset, \emptyset) & \lambda \in (W - Q) \cap (W \cap Q) = W \cap W' \cap Q \\ (\mathcal{L}_1 \cap \mathcal{B}_1, \mathcal{L}_2 \cap \mathcal{B}_2) & \lambda \in (W - Q) \cap (Q - W) = W' \cap W \cap Q \\ (\emptyset, \emptyset) & \lambda \in (W \cap Q) \cap (W - Q) = W \cap W' \cap Q' \\ ((\mathfrak{S}_1' \cap (\mathcal{L}_1' \cap \mathcal{B}_1)'), \mathfrak{S}_2' \cap (\mathcal{L}_2' \cap \mathcal{B}_2')) & \lambda \in (W \cap Q) \cap (W \cap Q) = W \cap W' \cap Q \end{cases}$$

Here, let's consider $W - Q$ in $R^{W \cup Q}$, since $W - Q = W \cap W'$, if an element is in the complement of W , then it is either in $Q - W$ or $(W \cup Q)'$. Thus, if $\lambda \in W - Q$, then either $\lambda \in W \cap W' \cap Q'$ or $\lambda \in W \cap W' \cap Q$. By taking into account this fact, $R^{W \cup Q} = W^{W \cup Q}$, where $W \cap W' \cap Q = W \cap W' \cap Q' = W \cap W' \cap Q = \emptyset$. It is obvious that $W \cap W' \cap Q = W \cap W' \cap Q'$ means that $W \cap (W \Delta Q) = \emptyset$.

Theorem 3.2.7 Let F^W, G^W , and H^Q be BSSs over U_1, U_2 . Then, the following distributions hold, where $W \cap W' \cap Q = (W \Delta W) \cap Q = \emptyset$;

- 1) $(F^W \tilde{U} G^W) \tilde{\theta}_\varepsilon H^Q = (F^W \tilde{\theta}_\varepsilon H^Q) \tilde{U} (G^W \tilde{\theta}_\varepsilon H^Q)$
- 2) $(F^W \tilde{\cap} G^W) \tilde{\theta}_\varepsilon H^Q = (F^W \tilde{\theta}_\varepsilon H^Q) \tilde{\cap} (G^W \tilde{\theta}_\varepsilon H^Q)$

Proof: (1) Let $F^W(\psi) = (\mathfrak{S}_1, \mathfrak{S}_2)$ for each $\psi \in W$, $G^W(\omega) = (\mathcal{L}_1, \mathcal{L}_2)$ for each $\omega \in W$, $H^Q(\varphi) = (\mathcal{B}_1, \mathcal{B}_2)$ for each $\varphi \in Q$ and $F^W \tilde{U} G^W = T^W$, where $\lambda \in W$,

$$T^W(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in W - W \\ ((\mathfrak{S}_1 \cup \mathcal{L}_1, \mathfrak{S}_2 \cup \mathcal{L}_2)) & \lambda \in W \cap W \end{cases}$$

Let $T^W \tilde{\theta}_\varepsilon H^Q = R^{W \cup Q}$, where $\lambda \in W \cup Q$,

$$R^{W \cup Q}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (W - W) - Q = W \cap W' \cap Q' \\ ((\mathfrak{S}_1 \cup \mathcal{L}_1, \mathfrak{S}_2 \cup \mathcal{L}_2)) & \lambda \in (W \cap W) - Q = W \cap W' \cap Q' \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in Q - W \\ ((\mathfrak{S}_1' \cap \mathcal{L}_1') \cap \mathcal{B}_1, (\mathfrak{S}_2' \cap \mathcal{L}_2') \cap \mathcal{B}_2) & \lambda \in (W - W) \cap Q = W \cap W' \cap Q \\ ((\mathfrak{S}_1' \cap \mathcal{L}_1') \cap \mathcal{B}_1', (\mathfrak{S}_2' \cap \mathcal{L}_2') \cap \mathcal{B}_2') & \lambda \in (W \cap W) \cap Q = W \cap W' \cap Q \end{cases}$$

Let's consider $(F^W \tilde{\theta}_\varepsilon H^Q) \tilde{U} (G^W \tilde{\theta}_\varepsilon H^Q)$. Let $F^W \tilde{\theta}_\varepsilon H^Q = K^{W \cup Q}$, where $\lambda \in W \cup Q$,

$$K^{W \cup Q}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in W - Q \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in Q - W \\ ((\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2')) & \lambda \in W \cap Q \end{cases}$$

Assume that $G^W \tilde{\theta}_\varepsilon H^Q = S^{W \cup Q}$, where $\lambda \in W \cup Q$,

$$S^{W \cup Q}(\lambda) = \begin{cases} (\mathcal{L}_1, \mathcal{L}_2) & \lambda \in W - Q \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in Q - W \\ ((\mathcal{L}_1' \cap \mathcal{B}_1', \mathcal{L}_2' \cap \mathcal{B}_2')) & \lambda \in W \cap Q \end{cases}$$

Let $K^{W \cup Q} \tilde{U} S^{W \cup Q} = W^{W \cup Q}$, where for each $\lambda \in W \cup Q$,

$$W^{W \cup Q}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (W - Q) - (W \cup Q) \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in (Q - W) - (W \cup Q) \\ ((\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2')) & \lambda \in (W \cap Q) - (W \cup Q) \\ ((\mathfrak{S}_1 \cup \mathcal{L}_1, \mathfrak{S}_2 \cup \mathcal{L}_2)) & \lambda \in (W - Q) \cap (W - Q) \\ ((\mathfrak{S}_1 \cup \mathcal{B}_1, \mathfrak{S}_2 \cup \mathcal{B}_2)) & \lambda \in (W - Q) \cap (Q - W) \\ ((\mathfrak{S}_1 \cup (\mathcal{L}_1' \cap \mathcal{B}_1'), \mathfrak{S}_2 \cap (\mathcal{L}_2' \cap \mathcal{B}_2')) & \lambda \in (W - Q) \cap (W \cap Q) \\ ((\mathcal{B}_1 \cup \mathcal{L}_1, \mathcal{B}_2 \cup \mathcal{L}_2)) & \lambda \in (Q - W) \cap (W - Q) \\ ((\mathcal{B}_1 \cup \mathcal{B}_1, \mathcal{B}_2 \cup \mathcal{B}_2)) & \lambda \in (Q - W) \cap (Q - W) \\ ((\mathcal{B}_1 \cup (\mathcal{L}_1' \cap \mathcal{B}_1'), \mathcal{B}_2 \cup (\mathcal{L}_2' \cap \mathcal{B}_2')) & \lambda \in (Q - W) \cap (W \cap Q) \\ ((\mathfrak{S}_1' \cap \mathcal{B}_1') \cup \mathcal{L}_1, (\mathfrak{S}_2' \cap \mathcal{B}_2') \cup \mathcal{L}_2) & \lambda \in (W \cap Q) \cap (W - Q) \\ ((\mathfrak{S}_1' \cap \mathcal{B}_1') \cup \mathcal{B}_1, (\mathfrak{S}_2' \cap \mathcal{B}_2') \cup \mathcal{B}_2) & \lambda \in (W \cap Q) \cap (Q - W) \\ ((\mathfrak{S}_1' \cap \mathcal{B}_1') \cup (\mathfrak{S}_2' \cap \mathcal{B}_2'), (\mathfrak{S}_2' \cap \mathcal{B}_2') \cup (\mathfrak{S}_2' \cap \mathcal{L}_2')) & \lambda \in (W \cap Q) \cap (W \cap Q) \end{cases}$$

Thus,

$$W^{W \cup Q}(\lambda) = \begin{cases} (\mathfrak{S}_1, \mathfrak{S}_2) & \lambda \in (W - Q) - (W \cup Q) = W \cap W' \cap Q' \\ ((\mathfrak{S}_1 \cup \mathcal{L}_1, \mathfrak{S}_2 \cup \mathcal{L}_2)) & \lambda \in (W - Q) \cap (W - Q) = W \cap W' \cap Q' \\ (\mathcal{B}_1, \mathcal{B}_2) & \lambda \in (Q - W) \cap (Q - W) = W' \cap W \cap Q \\ ((\mathcal{B}_1 \cup \mathcal{L}_1, \mathcal{B}_2 \cup \mathcal{L}_2)) & \lambda \in (Q - W) \cap (W \cap Q) = W' \cap W \cap Q \\ ((\mathfrak{S}_1' \cap \mathcal{B}_1', \mathfrak{S}_2' \cap \mathcal{B}_2')) & \lambda \in (W \cap Q) \cap (Q - W) = W \cap W' \cap Q' \\ ((\mathfrak{S}_1' \cap \mathcal{B}_1') \cup (\mathfrak{S}_2' \cap \mathcal{B}_2'), (\mathfrak{S}_2' \cap \mathcal{B}_2') \cup (\mathfrak{S}_2' \cap \mathcal{L}_2')) & \lambda \in (W \cap Q) \cap (W \cap Q) = W \cap W' \cap Q \end{cases}$$

Here, let's consider $Q - W$ in $R^{W \cup Q}$, since $Q - W = Q \cap W'$, if an element is in the complement of W , then it is either in $W - W$ or $(W \cup W)'$. Thus, if $\lambda \in Q - W$, then either $\lambda \in W' \cap W' \cap Q$ or $\lambda \in W' \cap W' \cap Q'$. By taking into account this fact, $R^{W \cup Q} = W^{W \cup Q}$, where $W' \cap W' \cap Q = W' \cap W' \cap Q' = W' \cap W' \cap Q = \emptyset$. It is obvious that $W' \cap W' \cap Q = W' \cap W' \cap Q' = \emptyset$ means that $(W \Delta W) \cap Q = \emptyset$.

Theorem 3.2.8 $(S^E(U_1, U_2), \tilde{\cap}, \tilde{\theta}_\varepsilon)$ and $(S^E(U_1, U_2), \tilde{U}, \tilde{\theta}_\varepsilon)$ are multiplicatively commutative and additively idempotent semirings with the identity \emptyset^\emptyset and without zero UCCs.

Proof. Soylu (2026) showed that $(S^E(U_1, U_2), \tilde{U})$ and $(S^V(U_1, U_2), \tilde{\cap})$ are non-commutative, idempotent monoids with the identity \emptyset_\emptyset , that is, a band (hence a semigroup). $(S^E(U_1, U_2), \tilde{\theta}_\varepsilon)$ is a commutative monoid (hence a semigroup) whose identity is \emptyset_\emptyset under the condition $W \cap W' \cap Q = \emptyset$, where F^W, G^W , and H^Q are BSSs over U_1, U_2 . Moreover, $\tilde{\theta}_\varepsilon$ distributes over \tilde{U} and $\tilde{\cap}$ under the conditions $W \cap W' \cap Q = (W \Delta W) \cap Q = W \cap (W \Delta Q) = \emptyset$. Consequently, UCCs, $(S^E(U_1, U_2), \tilde{U}, \tilde{\theta}_\varepsilon)$ and $(S^E(U_1, U_2), \tilde{\cap}, \tilde{\theta}_\varepsilon)$ are additive

idempotent multiplicatively commutative semirings upon reasonable request.
without zero, but with unity.

4. Conclusion

This work introduces a novel binary extended soft set operation for binary soft sets (BSSs). We aim to advance the BSS theory by proposing the concept of the "binary extended theta operation" of BSSs and closely analyzing the algebraic structures associated with this operation. The binary extended theta operation proposed in this study, instead of producing positive consensus, reveals common negative information by taking the intersection of the complements of binary soft sets on common parameters. In this respect, the theta operation provides a new algebraic perspective based on negative information for binary soft set theory and conceptually differs from existing binary extended operations in the literature. We also show that the collection of BSSs over the universe combined with binary extended theta operations and certain types of BSS operation forms various important algebraic structures under certain conditions. To expand this body of knowledge, future studies could examine different types of binary extended soft set operations as well as the corresponding distributions and their properties.

Authors' contribution

Aslihan Sezgin: Conceptualization, validation, formal analysis, data curation, supervision, Orhan Karamustafaoğlu: Methodology, software, validation, investigation, resources, data curation, writing—original draft preparation, writing—review and editing, visualization, supervision. All authors have read and agreed to the published version of the manuscript.

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Data availability statement

The data sets generated and/or analysed during the current study are included in this article. If more information is needed, it is available from the corresponding author

Conflicts of interest

The authors declare no conflicts of interest.

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