Decision Making and Analysis
https://ojs.luminescience.cn/DMA

LUMINESCIENCE
https://luminescience.cn

Original Research

Proper edge coloring of direct product of path and star fuzzy graphs

Bacha Khan[1], **Tabasam Rashid**[1], **Ismat Beg**[2*]

[1] University of Management and Technology, Lahore, Pakistan
[2] Lahore School of Economics, Lahore, Pakistan

[*] Corresponding to: Ismat Beg, Email: ibeg@lahoreschool.edu.pk

Abstract: This research explored adjacent vertex distinguishing proper edge coloring (AVDPEC) in the context of the direct product of fuzzy graphs, with particular emphasis on combinations of fuzzy path and fuzzy star graph. In this study, the classical coloring theory of crisp graphs were applied to fuzzy graphs, as membership values introduce an additional layer of intricacy to them. The main contribution of this research is an algorithm for the direct product of fuzzy path and fuzzy star networks, which acquires appropriate edge coloring that distinguishes adjacent vertices utilizing fuzzy membership values. In applications where recognizing between entities or tasks is crucial, this algorithm ensures that each adjacent vertex pair in the fuzzy graph product is identified by the specific colors of their incident edges. By using fuzzy membership values, we provided a versatile structure for managing complicated systems with incomplete or probabilistic interactions between nodes. We also introduced two significant applications of this edge coloring technique: load balancing in data networks and task parallelism in parallel computing. All things considered, our work strengthens the concept of AVDPEC in uncertain graphs, which benefits graph theory both hypothetically and essentially.

Keywords: Fuzzy path graphs, Fuzzy star graphs, Direct product of fuzzy graphs, Proper edge coloring, Fuzzy chromatic number

1. Introduction

Graph theory is a branch of mathematics that examines the associations and configurations of networks. A mathematical framework is utilized by graph theory to display numerous relationships between objects. The collections of edges and vertices are the two sets that structure a graph. The objects in a graph are addressed by the vertices, while the associations or networks amongst those objects are represented by the edges. Graph theory has been widely used across different disciplines to resolve real-world problems. For example, traffic networks can be demonstrated utilizing graphs where crossing points are addressed as vertices and roads as edges. However, classical graph theory, regardless of its broad use, has specific impediments while managing inadequate or dubious information. For example, the traffic congestion at an intersection is caused by various uncertain factors, including the quantity of vehicles, the cycle length of traffic signals, poor road conditions, and so on.

To address the limitations of classical graph theory, particularly in dealing with vulnerabilities, the concept of fuzzy sets was introduced by Zadeh [1] in 1965. Fuzzy set theory provides a quantitative framework for addressing

Received: Dec.15, 2025; Revised: Feb.15, 2026; Accepted: Mar.5, 2026; Published: May 15, 2026

DOI: https://doi.org/10.55976/dma.42026152224-42

vulnerability, vagueness, and uncertainty. When there are ambiguous boundaries amongst distinct states or classes, it provides a way to handle the situation. Instead of having a crisp binary value of 0 or 1, a fuzzy graph is an expansion of a classical graph in which each vertex or edge is linked with a degree of membership in the interval [0,1]. Fuzzy graphs were first proposed by Kaufmann [2], who developed a fundamental definition based on Zadeh's concept of fuzzy relations. In this manner, Rosenfeldand Mutab [3, 4] extended the definition of fuzzy graphs by introducing the idea of fuzzy vertices and fuzzy edges. Generally, a fuzzy graph can more effectively represent circumstances where the connections between objects are not rigorously binary but instead have fluctuating degrees of association. For instance, consider a social network where individuals are represented by vertices and the connections between them are represented by edges. In a classical graph, an edge either exists or does not, indicating the presence or absence of a relationship between two individuals. However, as a general rule, connections can have varying strengths, some individuals are close companions, while others are merely acquaintances. A fuzzy graph can capture these subtleties by relegating a membership degree to each edge, representing the strength of the relationship. For further information on fuzzy graph research, readers may consult [5-9].

In graph theory, graph coloring is a significant issue and it has significant hypothetical and viable importance. It involves assigning colors to certain elements of a graph, typically vertices or edges, subject to specific constraints. The purpose of graph coloring is to guarantee that neighboring or related components do not have a similar variety, thereby resolving conflicts in different real-world problems such as scheduling, frequency assignment, and resource allocation. In classical graph theory, the most essential kind of graph coloring is vertex coloring [10, 11], where colors are allocated to the vertices of a graph so that no two adjacent vertices share a similar color. This issue is known as proper vertex coloring. Similarly, proper edge coloring, local edge colouring and note on edge colouring [12-14] are specific types of graph coloring in which colors are allocated to the edges of a graph to such an extent that no two edges incident on a similar vertex share a similar color. The minimum number of colors required to color a graph G is named the chromatic number of G, represented by χ (G). Zhang [15, 16] presented the idea of adjacent strong edge coloring of graphs and adjacent strong edge colorings and total colorings of regular graph. Later on researchers [17-20], proposed the idea of AVDPEC guess in light of the conversation of the neighboring vertex differentiating proper edge chromatic numbers of trees, circles, complete bipartite graphs, and complete graphs. In which not only the edges must be properly colored, but for any pair of contiguous vertices u and v, the set of colors incident to u must be distinct from the set of colors incident to v. This type of coloring is particularly valuable in applications where adjacent vertices should be extraordinarily recognizable in view of the edges incident to them. The adjacent vertex differentiating proper edge chromatic number, denoted by χ_a (G), is the minimum possible number of colors expected to accomplish an AVDPEC of a graph G. Some basic operations of graphs and coloring discussed with some computations [21-24].

Considering that the coloring of fuzzy graph assumes an indispensable part in practical issues such as task and conflict coordination, an ever increasing number of researchers are dedicated to the investigation of fuzzy graph coloring issues. In classical graph theory, there are still a lot of coloring issues that have not yet tracked down comparing classifications in fuzzy graphs, such as the AVDPEC of the direct product of two fuzzy graphs. This gap will be filled through this study, which focuses on exploring the AVDPEC in the context of the direct product of fuzzy path and fuzzy star graphs. The edges of the direct product of fuzzy path and fuzzy star graphs are colored using a structured algorithm that guarantees proper edge coloring and satisfies the neighboring vertex differentiating constraint. In order to accomplish AVDPEC in fuzzy graphs, this research investigated a key approach. For the α-cut of G, the key technique relies on the advanced coloring AVDPEC function f_α of the crisp graph G_α (V, E_α). A subset of a fuzzy graph known as the α-cut arises when the edge membership values are greater than or equal to a certain threshold, α. The edges of the fuzzy graph are colored according to the AVDPEC criteria by gradually coloring the crisp graph G_α. An extension of the idea of the AVDPEC function through a space defined by colors, forms the groundwork of the subsequent approach. The support graph of the fuzzy graph, which takes into account the entire range of membership values rather than just the α-cut, serves as the groundwork for this extended coloring function.

Fuzzy graph theory has made significant advancements to address various levels of uncertainty and relationship pattern dynamics within the graph. In addition to traditional fuzzy graph theory, there are various extensions to these graph models. Fuzzy hypergraphs address relationship patterns among multiple nodes simultaneously, and then fuzzy directed graphs address the direction of relationship patterns. Fuzzy super hypergraphs that address relationship patterns among multiple nodes simultaneously for various vertex sets. A deeper definition for these membership patterns is found and they are discussed in intuitionistic fuzzy graphs, which consider both membership and non-membership in these graph structures. They take this graph representation to another level by introducing neutrosophic directed graphs, which include the degree of indeterminacy. More advanced graph representations include plithogenic graphs, which consider multiple attributes for these graph structures. Another advanced graph representation model is then quadripartitioned neutrosophic graph, which addresses multi-dimensional uncertainties for these graph structures. These various graph structure models adjust characteristics such as graph coloring relative to their membership, degree of indeterminacy, or various graph attribute constraints.

Much work has carried out by many researchers; for example, Fujita's contribution in hyper fuzzy sets and neutrosophic sets is remarkable [25, 26]. An application of extended plithogenic sets in the form of a plithogenic sociogram was proposed by Sudhs [27], which has opened the door for many researchers in this area. Distance measures of picture fuzzy sets and interval valued picture fuzzy sets were proposed by Zhu [28]. This study advances combinatorics in uncertain environments by employing graph and hypergraph structures, enriched with fuzzy, neutrosophic, soft set, rough set, and bipolar modeling approaches. For clarity in distinguishing between proposed concepts and conventional methods, we have listed the distinctions and improvements below. For instance, in conventional fuzzy graphs analogous to the proposed fuzzy graph concept, conditions concerning membership values are generally predetermined. The proposed concept, besides encompassing fuzzy graph structure, is capable of accommodating α-level structures. Furthermore, in the proposed concepts, vertex distinguishable proper edge coloring of conventional adjacency graphs is extended to fuzzy direct product graphs.

Table 1. Concise comparison among assumptions, limitations, extensions, and contributions of study

Existing concept	Main assumptions / limitations	Proposed concept / generalization	Main contributions of this paper
Fuzzy graphs	Edges have fixed membership; simple graphs only	Supports α-level structures and multi-edge interactions	Provides AVDPEC for direct product graphs under uncertainty
Intuitionistic fuzzy graphs	Membership and non-membership defined; limited to small graphs	Extends to direct product and path-star structures	Captures vertex distinguishing coloring under membership variability
Neutrosophic graphs	Truth, falsity, indeterminacy per edge; limited colorings	Includes α-level induced subgraphs for coloring	Derives supremum-based chromatic numbers reflecting uncertainty
Plithogenic graphs	Multi-attribute vertices; limited coloring approaches	Integrates multiple attributes into AVDPEC analysis	Bridges attribute-based uncertainty with structural coloring constraints

Table 1 presents a concise comparison highlighting the assumptions, limitations, extensions, and contributions of this work relative to the closest existing concepts. To better understand the relationship between fuzzy graphs and crisp graphs, α-cuts are the most effective tools for analysing the coordination. The chromatic invariant in fuzzy graph theory is the focus of this study. The emphasis shifts to distinguishing how uncertainty in edge participation can be molded through fuzzy membership values, which induces an entire group of structurally distinct α-level graphs whose coloring behavior needs to be analyzed in aggregate. This approach enables the researchers to assess chromatic requirements across a range of confidence levels – a capability that no single crisp realization can offer.

In the preliminary segment, the ADVPEC of fuzzy graphs based on the above approach will be discussed in detail, providing the foundation for applying the strategy to the direct product of fuzzy path and fuzzy star graphs. As a result, this research offers an in-depth technique for resolving the AVDPEC issue in fuzzy graphs, focusing on the direct product of fuzzy path and fuzzy star graphs. We will inspect two key applications that affect the AVDPEC structure in this exploration. The first is task parallelism in parallel computing, where the efficient allocation and execution of tasks on multiple processors require careful management of task dependencies. The second is load balancing in data networks, where distributing the load across network nodes while ensuring efficient communication and avoiding interference is a primary objective. The results presented in this study offer a foundation for further research in both graph theory and its applications in computing and networking.

In this proposed study, we have added different sections to help the reader better understand the sequence. Section 1 contains the introduction, and section 2 presents the literature review, basic and essential definitions, which are helpful to better understand the proposed work. In section 3, algorithm is added, in section 4 the algorithm's validity and useful results with explained figures are illustrated. Section 5 contains computational validity. Section 6 discusses real-life application and finally, section 7 describes the conclusion of the proposed study.

Motivation and novelty of the work

The theory of fuzzy graphs has been extensively used for modeling uncertain and incomplete information in graphs in the areas of communications, infrastructure planning, and decision-making processes. Various models and methods have been developed to study coloring properties and structural invariants of fuzzy graphs. Although the models of adjacent vertex distinguishing edge coloring in paths and cycles of fuzzy graphs has been studied earlier, less attention has been paid to the study on the behavior of this property in the construction of direct products of graphs. The aim of this research is to fill this research gap.

2. Definitions

This section will initiate by reviewing several definitions correlated to graph theory, fuzzy set theory, and fuzzy graph coloring, which will be utilized to define the chromatic number of fuzzy graphs. Subsequently, proper edge coloring for distinguishing neighboring vertices will be introduced for the direct product of fuzzy path and fuzzy star graphs.

Definition 1 [21]**.** A graph G = (V, E) is named a crisp graph with vertex set V =V (G) and edge set E = E (G). The number of vertices and edges in graph G are known as the order and size of G, represented by |V| and |E|, respectively.

Two vertices u and v are supposed to be adjacent if they are incident with a mutual edge (u, v). Two edges (u, v) and (u, w) are said to be contiguous if they share a common vertex u. An edge with identical endpoints is known as a loop, and two or more edges with a couple of vertices are called parallel edges. If an edge e in a graph G has no adjacent edges in G, then that edge e is known as an isolated edge.

Example 1: Let G = (V, E) be a crisp graph with vertex set V = {x_1, x_2, x_3, x_4, x_5, x_6} and edge set E = {e_1, e_2, e_3, e_4, e_5, e_6, e_7}, where e_1 = (x_1, x_2), e_2 = (x_2, x_3), e_3 = (x_3, x_4), e_4 = (x_4, x_2), e_5 = (x_4, x_2), e_6 = (x_4, x_1), e_7 = (x_5, x_6).

Crisp graph G is shown in Figure 1.

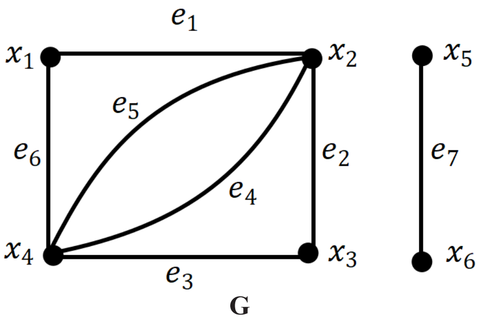


Figure 1. Crisp graph G

In the crisp graph G of Figure 1, e_4 and e_5 are parallel edges, while e_1, e_6 and e_2, e_3 are adjacent edges incident with common vertices x_1 and x_3, respectively. Since e_7 in graph G has no neighbors, hence it is an isolated edge in G.

Definition 2 [15]**.** A proper edge k-coloring of crisp graph G = (V, E) with a vertex set V (G) and an edge set E (G) is a mapping f from E to the set {1, 2, 3, …, k}, such that f (u, v) ≠ f (u, w), f (u, w) ≠ f (w, x) for all edges (u, v), (u, w), (w, x) ∈ E, whenever u, v, w, x ∈ V. The smallest k for which G acknowledges a proper edge k-coloring is known as the proper edge chromatic number of G represented by χ_a (G).

Example 2: Let a proper edge coloring of crisp graph G = (V, E) with a vertex set V = {x_1, x_2, x_3, x_4}, and an edge set E = {e_1, e_2, e_3, e_4} is a mapping f from E to the set of colors {red, green}, such that f (e_1) ≠ f (e_2), f (e_2) ≠ f (e_3), f (e_3) ≠ f (e_4) and f (e_4) ≠ f (e_2), where e_1 = (x_1, x_2), e_2 = (x_2, x_3), e_3 = (x_3, x_4), e_4 = (x_4, x_1).

Proper edge coloring of crisp graph G is shown in Figure 2.

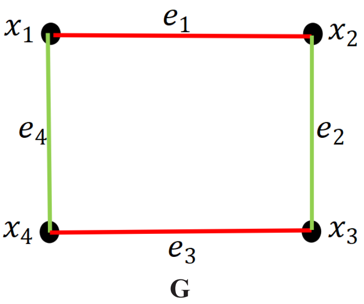


Figure 2. Proper edge coloring of crisp graph G

In the proper edge coloring of crisp graph G of Figure 2, e_1 and e_2 are adjacent edges incident with a common vertex x_2. Therefore, f (e_1)= red ≠ f (e_2)= green. Similarly, f (e_2)= green ≠ f (e_3)= red, f (e_3)= red ≠ f (e_4)= green, and f (e_4)= green ≠ f (e_1)= red. Now according to proper edge coloring, the least possible number of colors need to color the crisp graph G are exactly 2. Thus χ_a (G) = 2.

Definition 3 [18]**.** An AVDPEC is a type of proper edge coloring in which for any couple of neighboring vertices, the set of colors incident to u is not equivalent to the set of colors incident to v. In other words, given a graph G = (V, E), an AVDPEC is a function f: E → {1,2, 3,…, k}, such that f (u, v) ≠ f (u, w) ≠ f (w, x) for all edges (u, v), (u, w), (w, x) ∈ E. A k-AVDPEC is an AVDPEC utilizing maximum k colors. The least possible number of colors in an AVDPEC of G is known as the adjacent vertex differentiating proper edge chromatic number of G represented by χ_a (G).

Example 3: Let an AVDPEC of crisp graph G = (V, E) with a vertex set V = {x_1, x_2, x_3, x_4} and an edge set E = {e_1, e_2, e_3, e_4} is a mapping f from E to the set of colors {red, green, blue, yellow}, such that f (e_1) ≠ f (e_2) ≠ f (e_3), f (e_2) ≠ f (e_3) ≠ f (e_4), f (e_3) ≠ f (e_4) ≠ f (e_1), and f (e_4) ≠ f (e_1) ≠ f (e_2), where e_1 = (x_1, x_2), e_2 = (x_2, x_3), e_3 = (x_3, x_4), e_4 = (x_4, x_1).

AVDPEC of crisp graph G is shown in Figure 3.

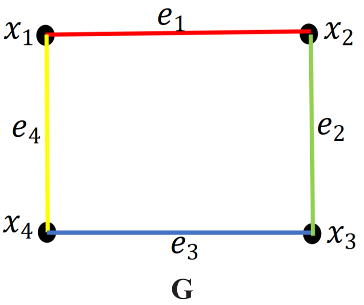


Figure 3. AVDPEC of crisp graph G

In the AVDPEC of crisp graph G of Figure 3, e_1,e_2 and e_2,e_3, are adjacent edges incident with common vertices x_2 and x_3, respectively. Therefore, according to AVDPEC, the collection of colors incident to x_2 is not equivalent to the collection of colors incident to x_3, because x_2 and x_3 are adjacent vertices. Thus f (e_1) = red ≠ f (e_2) = green ≠ f (e_3)

= blue. Similarly, f (e_2) = green ≠ f (e_3) = blue ≠ f (e_4) = yellow, f (e_3) = blue ≠ f (e_4) = yellow ≠ f (e_1) = red, and f (e_4) = yellow ≠ f (e_1)=red ≠ f (e_2)=green. Now according to AVDPEC, the least possible number of colors need to color the crisp graph G are 4. Thus χ_a (G) = 4.

Definition 4 [22]**.** A graph G = (V, E) is supposed to be a path graph P, if there exist an alternating sequence of different vertices $v_0, v_1, v_2, v_3, \ldots, v_n$ (except possibly v_0 and v_n), and edges $e_1, e_2, e_3, \ldots, e_n$, such that $e_i = (v_{i-1}, v_i)$, where $v_0, v_i \in V$, $e_i \in E$ and i=1, 2, 3, ..., n.

Example 4: Let an AVDPEC of path graph P_4 = (V, E) with a vertex set V = $\{x_1, x_2, x_3, x_4\}$, and an edge set E = $\{e_1, e_2, e_3\}$ is a mapping f from E to the set of colors {red, green, blue}, such that f (e_1) ≠ f (e_2) ≠ f (e_3), where $e_1 = (x_1, x_2)$, $e_2 = (x_2, x_3)$, $e_3 = (x_3, x_4)$.

AVDPEC of path graph P_4 is shown in Figure 4.

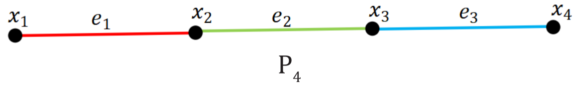


Figure 4. AVDPEC of path graph P_4

Definition 5 [23]**.** A graph G = (V, E) is supposed to be a star graph S, if there exist a fixed vertex u ∈ V, such that E = $\{(u, v_i) \mid v_i \in V \wedge u \neq v_i, i = 1, 2, 3, \ldots, n\}$. If the number of vertices in a star graph is n then it is known as *n*-star graph denoted by S_n. In *n*-star graph the degree of vertex is *n*-1 and is denoted by deg (n).

Example 5: Let an AVDPEC of star graph S_7 = (V, E) with a vertex set V = $\{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$ and an edge set E = $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is a mapping f from E to the set of colors{red, green, blue, yellow, pink, orange}, such that f (e_1) ≠ f (e_2) ≠ f (e_3) ≠ f (e_4) ≠ f (e_5) ≠ f (e_6), where $e_1 = (x_0, x_1)$, $e_2 = (x_0, x_2)$, $e_3 = (x_0, x_3)$, $e_4 = (x_0, x_4)$, $e_5 = (x_0, x_5)$, $e_6 = (x_0, x_6)$.

AVDPEC of star graph S_7 is shown in Figure 5.

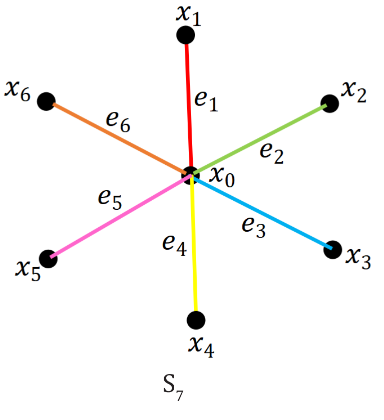


Figure 5. AVDPEC of star graph S_7

Definition 6 [24]**.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs such that $V_1 \cap V_2 = \emptyset$. The direct product $G_1 \sqcap G_2$ = (V, E) of graphs G_1 and G_2 is the graph with a vertex set $V = V_1 \times V_2$ and an edge set E = $\{((u_1, v_1), (u_2, v_2)) \mid (u_1, u_2) \in E_1\ (v_1, v_2) \in E_2\}$.

Example 6: Let an AVDPEC of path graph $P_4 = (V_1, E_1)$ with a vertex set $V_1 = \{x_1, x_2, x_3, x_4\}$, and an edge set $E_1 = \{e_1, e_2, e_3\}$ is a mapping f from E_1 to the set of colors {red, green, blue} such that f (e_1) ≠ f (e_2) ≠ f (e_3) and an AVDPEC of star graph $S_4 = (V_2, E_2)$ with a vertex set $V_2 = \{ y_1, y_2, y_3, y_4\}$ and an edge set $E_2 = \{l_1, l_2, l_3\}$ is a mapping g from E_2 to the set of colors {yellow, pink, orange} such that g (l_1) ≠ g (l_2) ≠ g (l_3), where $e_1 = x_1, x_2$, $e_2 = (x_2, x_3)$, $e_3 = (x_3, x_4)$ and $l_1 = (y_1, y_2)$, $l_2 = (y_1, y_3)$, $l_3 = (y_1, y_4)$. Then an AVDPEC of direct product $(P_4 \sqcap S_4)$ is shown in Figure 6.

In the AVDPEC of direct product $(P_4 \sqcap S_4)$, there are only two vertices (x_2, y_1) and (x_3, y_1), which have a maximum degree 6, but they are not adjacent. Therefore, according to AVDPEC, $(P_4 \sqcap S_4)$ needs exactly 6 different colors. Thus $\chi_a (P_4 \sqcap S_4) = 6$.

Definition 7 [1]**.** Let X be a nonempty set. A fuzzy set $\tilde{A}$ on X is well-defined as the family $\tilde{A} = \{(u, \mu_A (u)) \mid u \in X\}$, where μ_A: X → [0, 1] is the membership function and μ_A (u) is the membership degree of u to the fuzzy set $\tilde{A}$.

Definition 8 [4]**.** Let $\tilde{A}$ be a fuzzy set. A level set or α-cut is defined as $A_\alpha = \{ u \in X \mid \tilde{A} \geq \alpha\}$, where α ∈ [0,1]. If α = 1 then $A_1 = \{ u \in X \mid \tilde{A} = 1\}$ is known as the basic of the fuzzy set $\tilde{A}$. The support of the fuzzy set $\tilde{A}$ is defined as sup $\tilde{A} = \{u \in X \mid \tilde{A} > 0\}$.

Definition 9 [3]**.** A graph $\tilde{G}$ (V, σ, μ) is called a fuzzy graph with a non-empty set V along with a pair of functions σ: V → [0,1] and μ: V×V→ [0,1] such that ∀ x, y ∈ V, $\mu (x\ y) \leq \sigma (x) \wedge \sigma (y)$, where σ is the fuzzy vertex set of $\tilde{G}$ and μ is the fuzzy edge set of $\tilde{G}$, respectively.

Example 7: Let $\tilde{G}$ (V, σ, μ) be a fuzzy graph with a vertex set V = $\{x_1, x_2, x_3, x_4, x_5\}$. Define the fuzzy vertex set σ on V as σ (x_1)=0.4, σ (x_2)=0.3, σ (x_3)=0.5, σ (x_4)=1, σ (x_5) = 0.7. Also define the fuzzy edge set μ on V × V such that μ (x_1x_2) = 0.2, μ (x_2x_3) = 0.3, μ (x_3x_4) = 0.4, μ (x_4x_1) = 0.1, μ (x_1x_3) = 0.3, and μ (x_3x_5) = 0.5.

Fuzzy graph $\tilde{G}$ is shown in Figure 7.

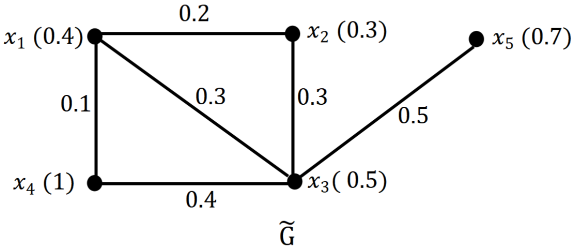


Figure 7. Fuzzy graph $\tilde{G}$

This study explored a fuzzy graph where σ (u) = 1 for every u in V. $\tilde{G}$ (V, $\tilde{E}$) can consequently be utilized to address a fuzzy graph.

Definition 10 [3]**.** Let $\tilde{G}$ (V, $\tilde{E}$) be a fuzzy graph. A path $\tilde{P}$ in a fuzzy graph $\tilde{G}$ is an order of different vertices $v_0, v_1, v_2, v_3, \ldots, v_n$ (except possibly v_0 and v_n) such that $\mu (v_{i-1}, v_i) > 0$ where $0 < i \leq n$.

Definition 11 [3]**.** A fuzzy graph $\tilde{G}$ (V, $\tilde{E}$) is called a fuzzy star graph $\tilde{S}$ if there exists only one fixed vertex u ∈ V such that $\mu_{\tilde{E}} (u\ v_i) > 0$ and $\mu_{\tilde{E}} (v_i\ v_j) = 0$ for all $v_i, v_j \in V\text{-}\{u\}$.

Definition 12 [5]**.** Let $\tilde{G}_1 (V_1, \tilde{E}_1)$ and $\tilde{G}_2 (V_2, \tilde{E}_2)$ be two fuzzy graphs such that $V_1 \cap V_2 = \emptyset$. The theorem $\widetilde{G}_1 \Pi \widetilde{G}_2$ $(V, \widetilde{E})$ of fuzzy graphs $\widetilde{G}_1$ and $\widetilde{G}_2$ is a fuzzy graph with a vertex set $V = V_1 \times V_2$ and fuzzy edge set $\widetilde{E} = \{((u_1, v_1), (u_2, v_2)) \mid (u_1, u_2) \in \widetilde{E}_1\ (v_1, v_2) \in \widetilde{E}_2\}$. The membership values of vertices and edges in the fuzzy graph $\widetilde{G}_1 \Pi \widetilde{G}_2$ are given by

$(\sigma_1 \Pi \sigma_2)(x, y) = \sigma_1(x) \wedge \sigma_2(y) \quad \forall\, x, y \in V$ and

$(\mu_1 \Pi \mu_2)((a_1, b_1), a_2, b_2) = \mu_1(a_1, a_2) \wedge \mu_2(b_1, b_2) \quad \forall\, (a_1, a_2), (b_1, b_2) \in \widetilde{E}$.

Definition 13 [25]**.** Let $\widetilde{G}$ $(V, \widetilde{E})$ be a fuzzy graph and $G_\alpha = \{(V, E_\alpha) \mid \alpha \in [0,1]\}$ be the family of α-cut set of fuzzy graph $\widetilde{G}$. An AVDPEC of fuzzy graph $\widetilde{G}$ is a mapping $\{f_\alpha : E_\alpha \rightarrow \{1, 2, 3, \ldots, k\} \mid \alpha \in [0,1]\}$ such that f_α fulfilled the AVDPEC of crisp graph G = (V, E). A k-AVDPEC is an AVDPEC utilizing maximum k colors and an adjacent vertex differentiating proper edge chromatic number of fuzzy graph $\widetilde{G}$ is $\chi_a^\alpha(\widetilde{G}) = \sup\{\chi_a(G_\alpha) \mid \alpha \in [0,1]\}$.

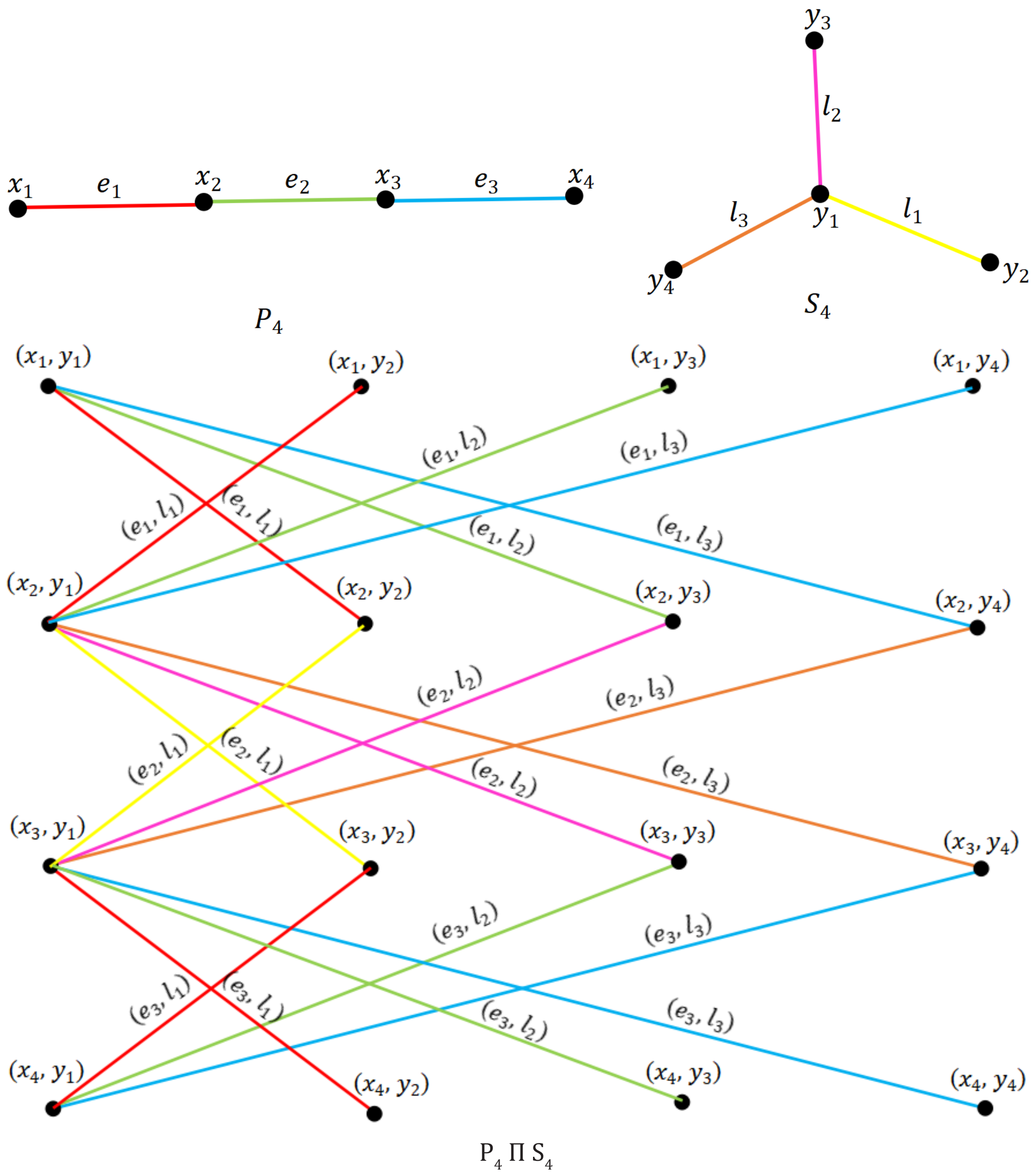


Figure 6. AVDPEC of direct product ($P_4 \Pi S_4$)

3. AVDPEC coloring procedure for fuzzy star and path direct products

The algorithm is designed to maintain validity for all possible admissible α-level realizations of a fuzzy graph. The process of assigning colors to the direct product of two fuzzy graphs is as follows:

Firstly, for all $\alpha \in [0,1]$ of edges in $\tilde{P}_m$ or $\tilde{S}_n$, assign the minimum color to the edge in $\tilde{P}_m \Pi \tilde{S}_n$ and choose the color of the edge with the minimum membership value incident with vertices of $\tilde{P}_m$ and $\tilde{S}_n$. Assign that color to the edge in $\tilde{P}_m \Pi \tilde{S}_n$, then assign colors to the edges already used in $\tilde{P}_m \Pi \tilde{S}_n$. Finally, assign new colors to the edges in $\tilde{P}_m \Pi \tilde{S}_n$ if necessary.

4. Algorithms validity

At each step, the algorithm uses a color different from those of the edges at adjacent vertices, thus achieving its properness. Moreover, a selection rule explicitly excludes assignment choices that would result in the incident points of an edge having the same incident colors. Attribution [2] is one of the earliest works to study constraints on graph coloring. Specifically this aspect. Because these are maintained during the entire process, it is guaranteed that the final coloring is an example of an adjacent vertex distinguishing proper edge coloring of the α-cut graph.

Remark: For the class of fuzzy direct product graphs studied here, the number of colors generated by the algorithm aligns with the bounds derived in the theoretical analysis, reflecting coherence between the algorithmic procedure and the analytical findings.

Example 8: Let an AVDPEC of fuzzy path graph $\tilde{P}_5$ (V_1, $\tilde{E}_1$) with a vertex set $V_1 = \{x_1, x_2, x_3, x_4, x_5\}$ and fuzzy edge set μ on $\tilde{E}_1$ is a mapping f: $\tilde{E}_1 \rightarrow$ {red, green, blue} such that $f(x_1, x_2) \neq f(x_2, x_3) \neq f(x_3, x_4)$, $f(x_2, x_3) \neq f(x_3, x_4) \neq f(x_4, x_5)$ and an AVDPEC of fuzzy star graph $\tilde{S}_5$ (V_2, $\tilde{E}_2$) with a vertex set $V_2 = \{y_1, y_2, y_3, y_4, y_5\}$ and fuzzy edge set μ on $\tilde{E}_2$ is a mapping g: $\tilde{E}_2 \rightarrow$ {red, green, blue, yellow} such that $g(y_1, y_2) \neq g(y_1, y_3) \neq g(y_1, y_4) \neq g(y_1, y_5)$ where $\mu_{\tilde{E}_1}(x_1\,x_2) = 0.1$, $\mu_{\tilde{E}_1}(x_2\,x_3) = 0.3$, $\mu_{\tilde{E}_1}(x_3\,x_4) = 0.4$, $\mu_{\tilde{E}_1}(x_4\,x_5) = 0.2$, $\mu_{\tilde{E}_2}(y_1\,y_2) = 0.2$, $\mu_{\tilde{E}_2}(y_1\,y_3) = 0.4$, $\mu_{\tilde{E}_2}(y_1\,y_4) = 0.3$ and $\mu_{\tilde{E}_2}(y_1\,y_5) = 0.1$. Then an AVDPEC of direct product ($\tilde{P}_5 \Pi \tilde{S}_5$) is shown in Figure 8.

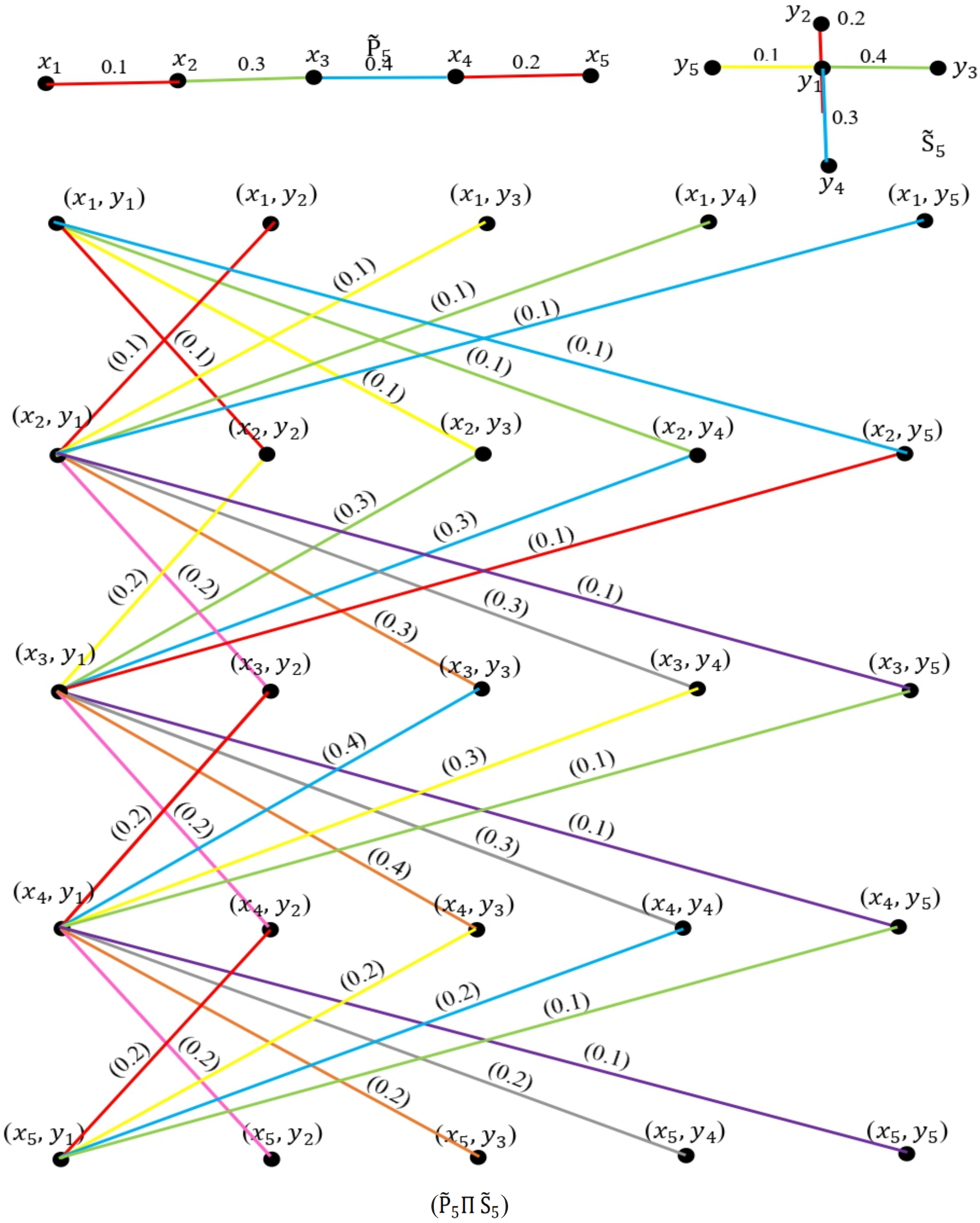
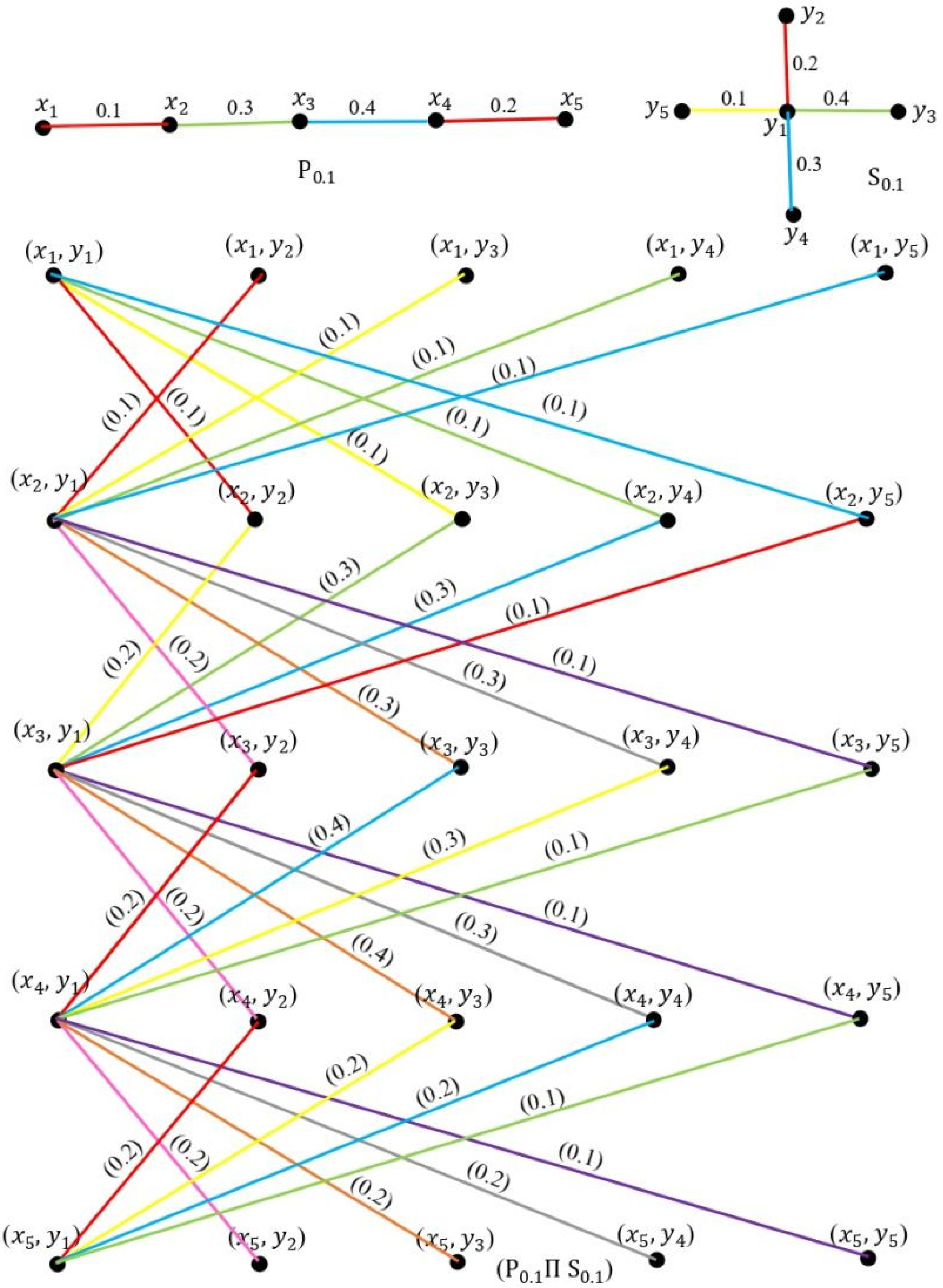
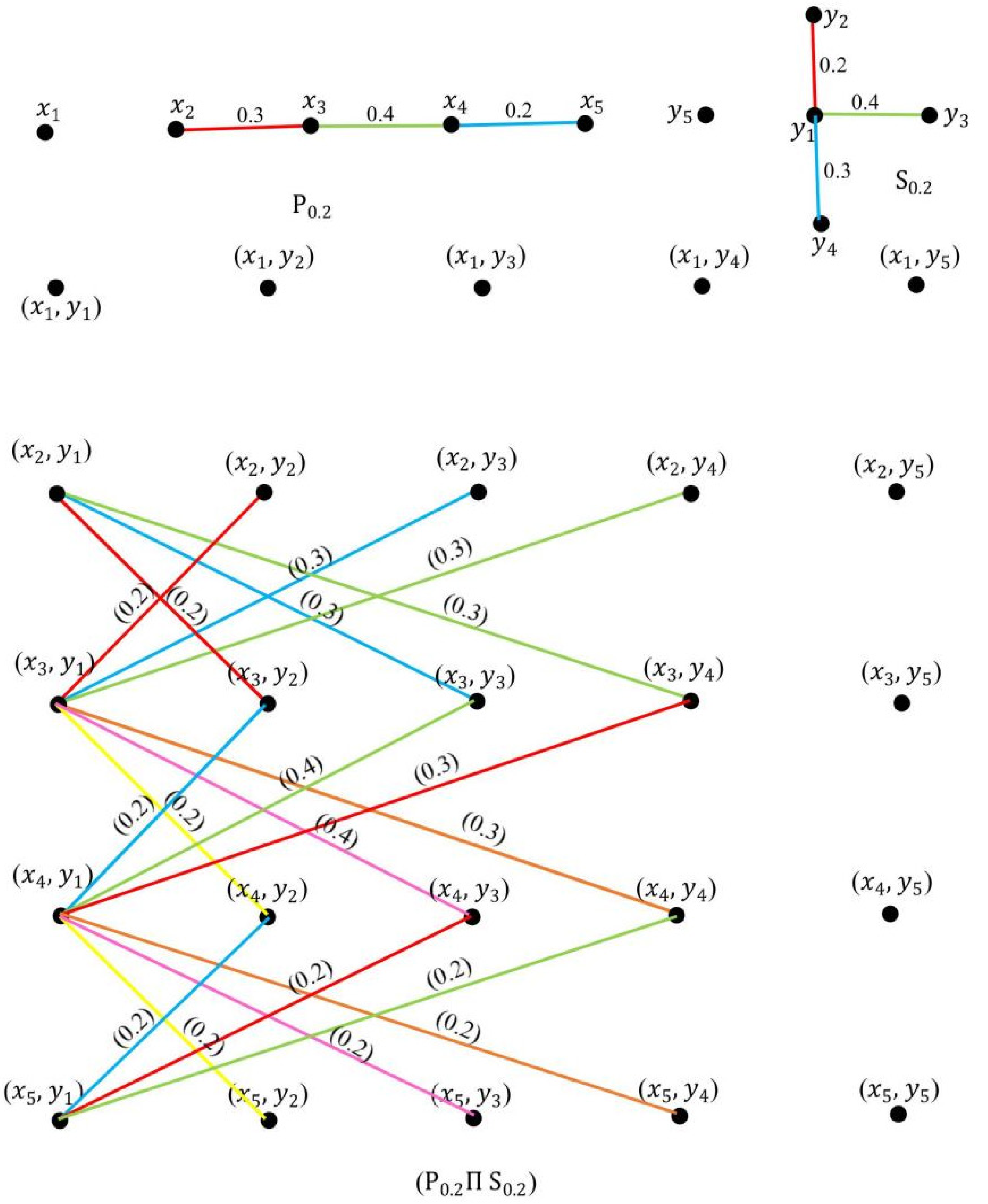


Figure 8. AVDPEC of direct product $\tilde{P}_5 \Pi \tilde{S}_5$

In the AVDPEC of direct product ($\tilde{P}_5 \Pi \tilde{S}_5$) five α-cut graphs of P_α (V_1, $E_{1\alpha}$), S_α (V_2, $E_{2\alpha}$) and ($P_\alpha \Pi S_\alpha$) (V, E_α) are acquired by taking into account the values of $\alpha \in [0,1]$. AVDPEC of five α-cut graphs of P_α, S_α and $P_\alpha \Pi S_\alpha$ are shown in Figure 9.

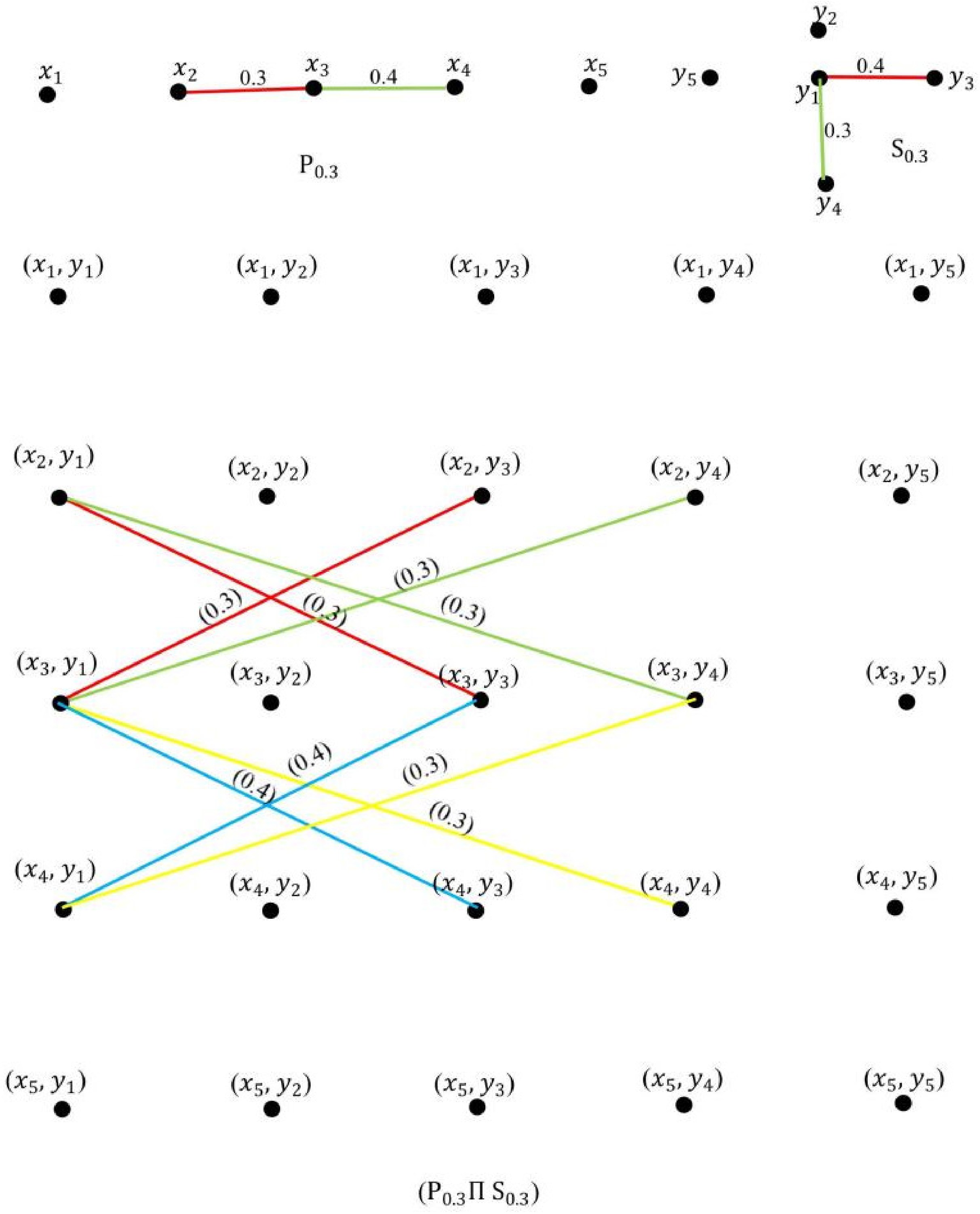


(a). for α = 0.1

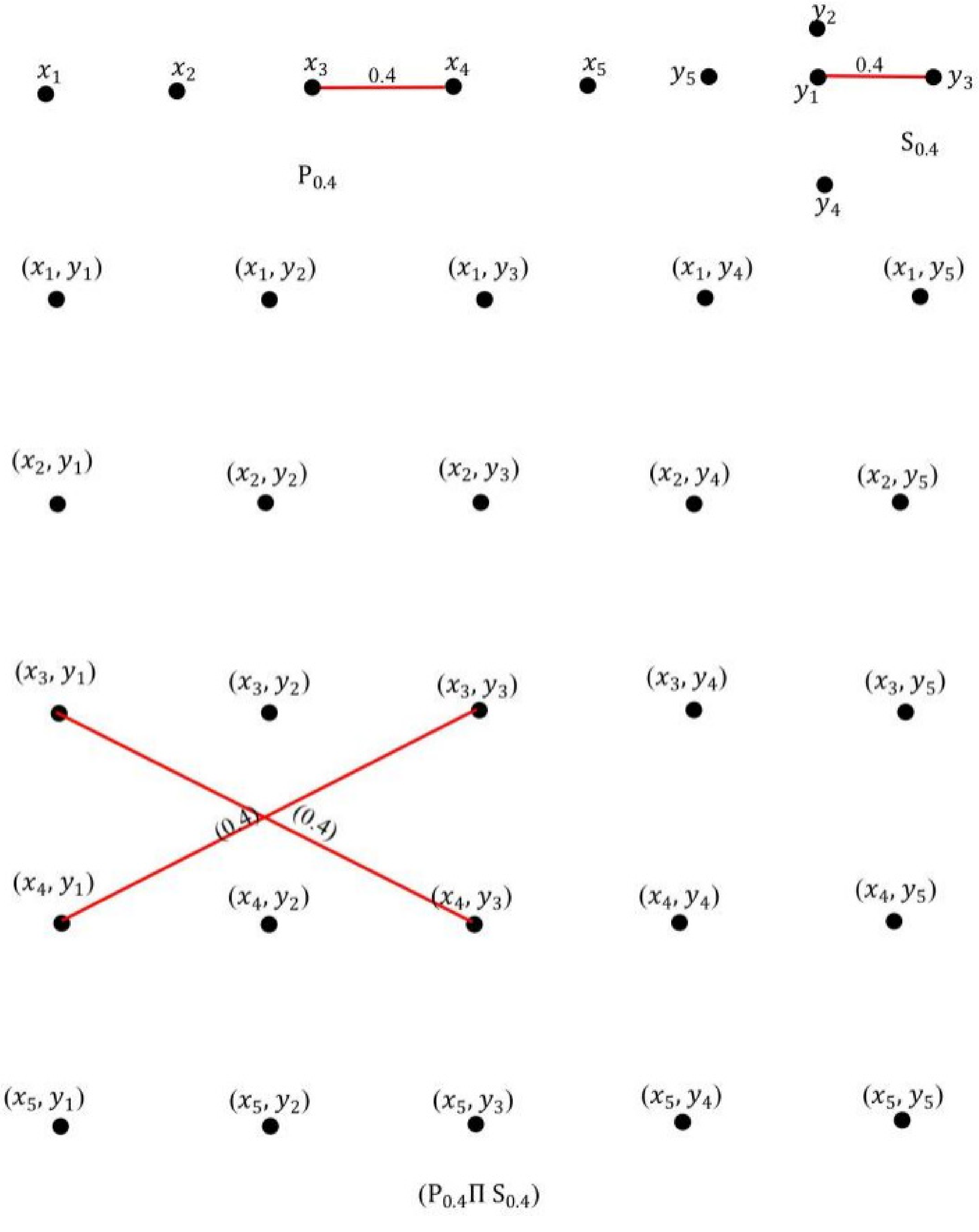


x_1
x_2
0.3
x_3
0.4
x_4
0.2
x_5
$P_{0.2}$
y_5
y_2
0.2
0.4
y_3
y_1
0.3
y_4
$S_{0.2}$
(x_1, y_1)
(x_1, y_2)
(x_1, y_3)
(x_1, y_4)
(x_1, y_5)
(x_2, y_1)
(x_2, y_2)
(x_2, y_3)
(x_2, y_4)
(x_2, y_5)
(0.3)
(0.3)
(0.2)
(0.2)
(0.3)
(0.3)
(x_3, y_1)
(x_3, y_2)
(x_3, y_3)
(x_3, y_4)
(x_3, y_5)
(0.4)
(0.3)
(0.2)
(0.2)
(0.4)
(0.3)
(x_4, y_1)
(x_4, y_2)
(x_4, y_3)
(x_4, y_4)
(x_4, y_5)
(0.2)
(0.2)
(0.2)
(0.2)
(0.2)
(0.2)
(x_5, y_1)
(x_5, y_2)
(x_5, y_3)
(x_5, y_4)
(x_5, y_5)
$(P_{0.2} \Pi S_{0.2})$

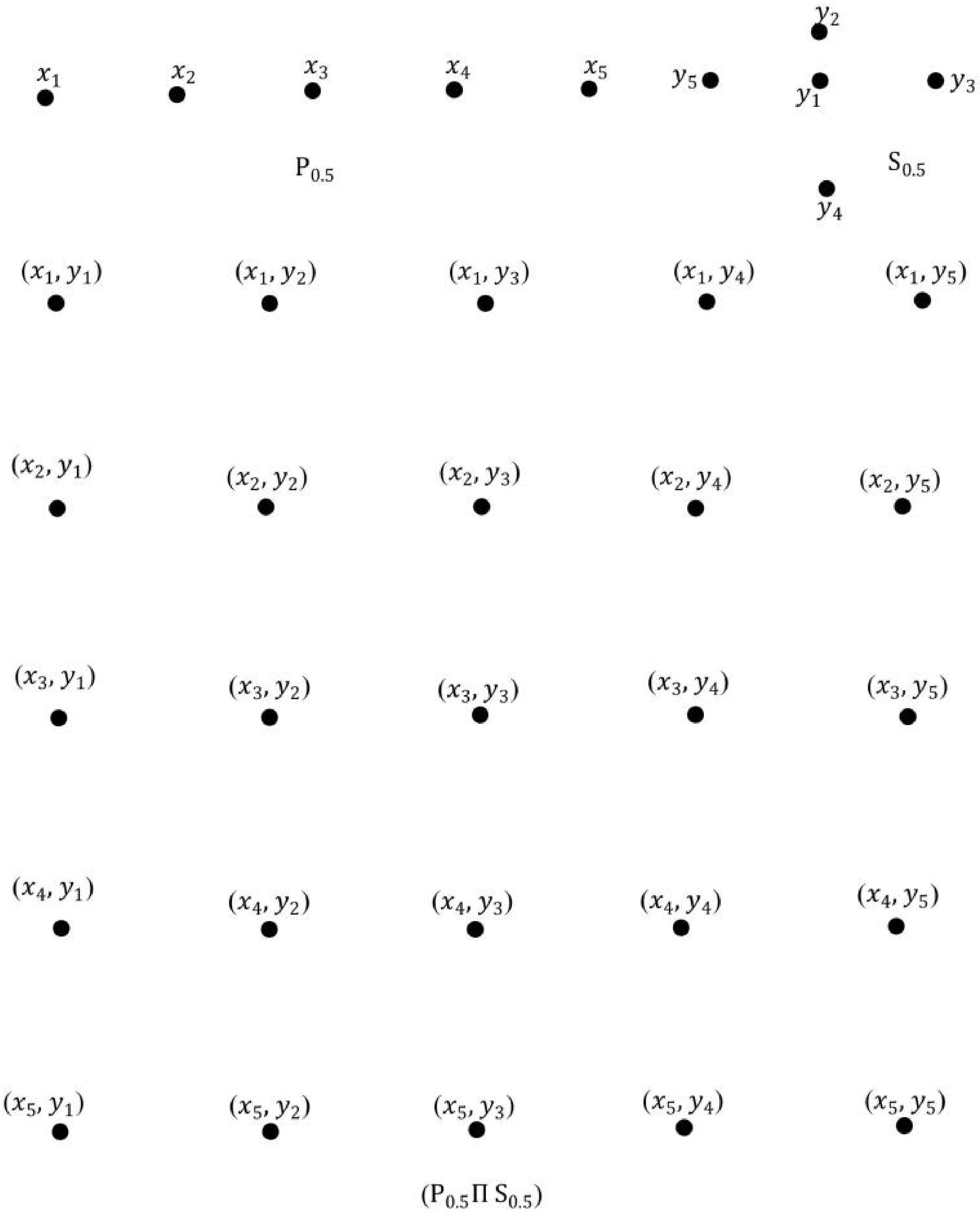
(b). for α = 0.2


x_1
x_2
0.3
x_3
0.4
x_4
x_5
y_5
y_2
y_1
0.4
y_3
0.3
y_4
$P_{0.3}$
$S_{0.3}$
(x_1, y_1)
(x_1, y_2)
(x_1, y_3)
(x_1, y_4)
(x_1, y_5)
(x_2, y_1)
(x_2, y_2)
(x_2, y_3)
(x_2, y_4)
(x_2, y_5)
(0.3)
(0.3)
(0.3)
(0.3)
(x_3, y_1)
(x_3, y_2)
(x_3, y_3)
(x_3, y_4)
(x_3, y_5)
(0.4)
(0.4)
(0.3)
(0.3)
(x_4, y_1)
(x_4, y_2)
(x_4, y_3)
(x_4, y_4)
(x_4, y_5)
(x_5, y_1)
(x_5, y_2)
(x_5, y_3)
(x_5, y_4)
(x_5, y_5)
$(P_{0.3} \Pi S_{0.3})$

(c). for α = 0.3



(d). for $\alpha = 0.4$



(e). for α = 0.5

Figure 9. AVDPEC of five α-cut graphs $P_\alpha \Pi S_\alpha$; (a). for α = 0.1; (b). for α = 0.2; (c). for α = 0.3; (d). for α = 0.4; (e). for α = 0.5

For each α ∈ [0,1] the neighboring vertex differentiating proper edge chromatic number $\chi_a(P_\alpha)$, $\chi_a(S_\alpha)$, $\chi_a(P_\alpha \Pi S_\alpha)$ and the set of colors incident to vertices x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5, (x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), (x_1, y_5), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_2, y_5), (x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), (x_3, y_5), (x_4, y_1), (x_4, y_2), (x_4, y_3), (x_4, y_4), (x_4, y_5), (x_5, y_1), (x_5, y_2), (x_5, y_3), (x_5, y_4) and (x_5, y_5) are shown in Table 2, 3 and 4, respectively.

Table 2. The adjacent vertex differentiating proper chromatic number edge

α	P_α	$\chi_a(P_\alpha)$	$I_f^\alpha(x_1)$	$I_f^\alpha(x_2)$	$I_f^\alpha(x_3)$	$I_f^\alpha(x_4)$	$I_f^\alpha(x_5)$
0.1	$P_{0.1}$	3	{R}	{R,G}	{G,B}	{B,R}	{R}
0.2	$P_{0.2}$	3	{ }	{R}	{R,G}	{G,B}	{B}
0.3	$P_{0.3}$	2	{ }	{R}	{R,G}	{G}	{ }
0.4	$P_{0.4}$	1	{ }	{ }	{R}	{R}	{ }
0.5	$P_{0.5}$	0	{ }	{ }	{ }	{ }	{ }

Table 3. The neighboring vertex differentiating proper edge chromatic number

α	S_α	$\chi_a(S_\alpha)$	$I_f^\alpha(y_1)$	$I_f^\alpha(y_2)$	$I_f^\alpha(y_3)$	$I_f^\alpha(y_4)$	$I_f^\alpha(y_5)$
0.1	$S_{0.1}$	4	{R,G,B,Y}	{R}	{G}	{B}	{Y}
0.2	$S_{0.2}$	3	{R,G,B}	{R}	{G}	{B}	{ }
0.3	$S_{0.3}$	2	{R,G}	{ }	{R}	{G}	{ }
0.4	$S_{0.4}$	1	{R}	{ }	{R}	{ }	{ }
0.5	$S_{0.5}$	0	{ }	{ }	{ }	{ }	{ }

Table 4. The neighboring edge differentiating proper edge chromatic number

A	$\mathbf{P_\alpha \Pi S_\alpha}$	$\chi_a(\mathbf{P_\alpha \Pi S_\alpha})$	$I_f^\alpha(x_1, y_1)$	$I_f^\alpha(x_1, y_2)$	$I_f^\alpha(x_1, y_3)$	$I_f^\alpha(x_1, y_4)$	$I_f^\alpha(x_1, y_5)$
0.1	$P_{0.1} \Pi S_{0.1}$	8	{R,G,B,Y}	{R}	{Y}	{G}	{B}
0.2	$P_{0.2} \Pi S_{0.2}$	6	{ }	{ }	{ }	{ }	{ }
0.3	$P_{0.3} \Pi S_{0.3}$	4	{ }	{ }	{ }	{ }	{ }
0.4	$P_{0.4} \Pi S_{0.4}$	1	{ }	{ }	{ }	{ }	{ }
0.5	$P_{0.5} \Pi S_{0.5}$	0	{ }	{ }	{ }	{ }	{ }
A	$\mathbf{P_\alpha \Pi S_\alpha}$	$\chi_a(\mathbf{P_\alpha \Pi S_\alpha})$	$I_f^\alpha(x_2, y_1)$	$I_f^\alpha(x_2, y_2)$	$I_f^\alpha(x_2, y_3)$	$I_f^\alpha(x, y_4)$	$I_f^\alpha(x_2, y_5)$
0.1	$P_{0.1} \Pi S_{0.1}$	8	{R,G,B,Y, P,O,G,P}	{R,Y}	{Y,G}	{G,B}	{R,B}
0.2	$P_{0.2} \Pi S_{0.2}$	6	{R,G,B}	{R}	{B}	{G}	{ }
0.3	$P_{0.3} \Pi S_{0.3}$	4	{R,G}	{ }	{R}	{G}	{ }
0.4	$P_{0.4} \Pi S_{0.4}$	1	{ }	{ }	{ }	{ }	{ }
0.5	$P_{0.5} \Pi S_{0.5}$	0	{ }	{ }	{ }	{ }	{ }
A	$\mathbf{P_\alpha \Pi S_\alpha}$	$\chi_a(\mathbf{P_\alpha \Pi S_\alpha})$	$I_f^\alpha(x_3, y_1)$	$I_f^\alpha(x_3, y_2)$	$I_f^\alpha(x_3, y_3)$	$I_f^\alpha(x_3, y_4)$	$I_f^\alpha(x_3, y_5)$
0.1	$P_{0.1} \Pi S_{0.1}$	8	{R,G,B,Y, P,O}	{R,P}	{B,O}	{G,Y}	{G,P}
0.2	$P_{0.2} \Pi S_{0.2}$	6	{R,G,B,Y, P,O}	{R,B}	{B,G}	{R,G}	{ }
0.3	$P_{0.3} \Pi S_{0.3}$	4	{R,G,B,Y}	{ }	{R,B}	{G,Y}	{ }
0.4	$P_{0.4} \Pi S_{0.4}$	1	{R}	{ }	{R}	{ }	{ }
0.5	$P_{0.5} \Pi S_{0.5}$	0	{ }	{ }	{ }	{ }	{ }

A	$P_\alpha \Pi S_\alpha$	$\chi_a (P_\alpha \Pi S_\alpha)$	$I_f^\alpha (x_4, y_1)$	$I_f^\alpha (x_4, y_2)$	$I_f^\alpha (x_4, y_3)$	$I_f^\alpha (x_4, y_4)$	$I_f^\alpha (x_4, y_5)$
0.1	$P_{0.1} \Pi S_{0.1}$	8	{R,G,B,Y, P,O,G,P}	{R,P}	{O,Y}	{G,B}	{G,P}
0.2	$P_{0.2} \Pi S_{0.2}$	6	{R,G,B,Y, P,O}	{B,Y}	{R,P}	{G,O}	{ }
0.3	$P_{0.3} \Pi S_{0.3}$	4	{B,Y}	{ }	{B}	{Y}	{ }
0.4	$P_{0.4} \Pi S_{0.4}$	1	{R}	{ }	{R}	{ }	{ }
0.5	$P_{0.5} \Pi S_{0.5}$	0	{ }	{ }	{ }	{ }	{ }
A	$P_\alpha \Pi S_\alpha$	$\chi_a (P_\alpha \Pi S_\alpha)$	$I_f^\alpha (x_5, y_1)$	$I_f^\alpha (x_5, y_2)$	$I_f^\alpha (x_5, y_3)$	$I_f^\alpha (x_5, y_4)$	$I_f^\alpha (x_5, y_5)$
0.1	$P_{0.1} \Pi S_{0.1}$	8	{R,G,B,Y}	{P}	{O}	{G}	{P}
0.2	$P_{0.2} \Pi S_{0.2}$	6	{R,G,B}	{Y}	{P}	{O}	{ }
0.3	$P_{0.3} \Pi S_{0.3}$	4	{ }	{ }	{ }	{ }	{ }
0.4	$P_{0.4} \Pi S_{0.4}$	1	{ }	{ }	{ }	{ }	{ }
0.5	$P_{0.5} \Pi S_{0.5}$	0	{ }	{ }	{ }	{ }	{ }

In Table 2, the adjacent vertex differentiating proper chromatic number edge of 0.1-cut graph $P_{0.1}$ is 3. Therefore, $\chi_a (P_{0.1}) = 3$. Similarly, $\chi_a (P_{0.2}) = 3$, $\chi_a (P_{0.3}) = 2$, $\chi_a (P_{0.4}) = 1$ and $\chi_a (P_{0.5}) = 0$. Thus $\chi_a^\alpha (\tilde{P}_5) = \sup \{3, 3, 2, 1, 0\} = 3$.

In Table 3, the neighboring vertex differentiating proper edge chromatic number of 0.1-cut graph $S_{0.1}$ is 4. Therefore, $\chi_a (S_{0.1}) = 4$. Similarly, $\chi_a (S_{0.2}) = 3$, $\chi_a (S_{0.3}) = 2$, $\chi_a (S_{0.4}) = 1$ and $\chi_a (S_{0.5}) = 0$. Thus $\chi_a^\alpha (\tilde{S}_5) = \sup \{4, 3, 2, 1, 0\} = 4$.

In Table 4, the neighboring vertex differentiating proper edge chromatic number of 0.1-cut graph $P_{0.1} \Pi S_{0.1}$ is 8. Therefore, $\chi_a (P_{0.1} \Pi S_{0.1}) = 8$. Similarly, $\chi_a (P_{0.2} \Pi S_{0.2}) = 6$, $\chi_a (P_{0.3} \Pi S_{0.3}) = 4$, $\chi_a (P_{0.4} \Pi S_{0.4}) = 1$ and $\chi_a (P_{0.5} \Pi S_{0.5}) = 0$. Thus $\chi_a^\alpha (\tilde{P}_5 \Pi \tilde{S}_5) = \sup \{8, 6, 4, 1, 0\} = 8$.

Lemma 1. If P_n is a path graph of order $n \leq 3$ and S_2 is a star graph of order 2 then,

$$\chi_a (P_n \Pi S_2) = \begin{cases} 2, & n = 3 \\ 1, & n = 2 \end{cases}$$

The following theorem is given with respect to adjacent vertex. The adjacent vertex distinguishing the proper edge chromatic number of a direct product of a fuzzy path and a fuzzy star graph is determined by analyzing the evolution of degree configurations across α-level induced subgraphs arising from variations in membership values over the edges.

Theorem 2. Let $\tilde{P}_m$ be a fuzzy path graph of order $m \geq 4$ and $\tilde{S}_n$ be a fuzzy star graph of order $n \geq 3$. Show the following cases for any $\alpha \in [0,1]$,

I. If $\chi_a^\alpha (\tilde{P}_m) = 3$ and $\chi_a^\alpha (\tilde{S}_n) = n - 2$

Then $\chi_a^\alpha (\tilde{P}_m \Pi \tilde{S}_n) = 2\,(n - 2)$

II. If $\chi_a^\alpha (\tilde{P}_m) = 1$ and $\chi_a^\alpha (\tilde{S}_n) = n - 1$

Then $\chi_a^\alpha (\tilde{P}_m \Pi \tilde{S}_n) = n - 1$

III. If $\chi_a^\alpha (\tilde{P}_m) = 2$ and $\chi_a^\alpha (\tilde{S}_n) = n - 1$

Then $\chi_a^\alpha (\tilde{P}_m \Pi \tilde{S}_n) = 2\,(n - 1)$

IV. If $\chi_a^\alpha (\tilde{P}_m) = 2$ and $\chi_a^\alpha (\tilde{S}_n) = n - 2$

Then $\chi_a^\alpha (\tilde{P}_m \Pi \tilde{S}_n) = 2\,(n - 2)$

V. If $\chi_a^\alpha (\tilde{P}_m) = 1$ and $\chi_a^\alpha (\tilde{S}_n) = n - k$

Then $\chi_a^\alpha (\tilde{P}_m \Pi \tilde{S}_n) = n - k$ for some $k \geq 0$

Proof:

Let us consider $\tilde{P}_m (V_1, \sigma_1, \mu_1)$ be a fuzzy path graph of order $m \geq 4$ and $\tilde{S}_n (V_2, \sigma_2, \mu_2)$ be a fuzzy star graph of order $n \geq 3$ such that

$\sigma_1(v), \mu_1(e) \in [0, 1] \quad \forall v \in V_1 (\tilde{P}_m), e \in E_1 (\tilde{P}_m)$

$\sigma_2(v), \mu_2(e) \in [0, 1] \quad \forall v \in V_2 (\tilde{S}_n), e \in E_2 (\tilde{S}_n)$

As $\chi_a (\tilde{P}_m) = 3$ where $m \geq 4$

And $\chi_a (\tilde{S}_n) = n-1$ where $n \geq 3$

Now by definition [2]

$V = V_1 \times V_2$ and $\widetilde{E} = \{(x_1, y_1), (x_2, y_2) \mid (x_1, x_2) \in \widetilde{E}_1 (y_1, y_2) \in \widetilde{E}_2\}$

Where

$(\sigma_1 \Pi \sigma_2) (x, y) = \sigma_1(x) \wedge \sigma_2(y) \quad \forall x, y \in V$ and

$(\mu_1 \Pi \mu_2) ((x_1, y_1), x_2, y_2) = \mu_1(x_1, x_2) \wedge \mu_2(y_1, y_2) \quad \forall (x_1, x_2), (y_1, y_2) \in \widetilde{E}$

Since $\tilde{P}_m \Pi \tilde{S}_n \in 2\,(n-1)$.

And $\chi_a (\tilde{P}_m \Pi \tilde{S}_n) = 2\,(n-1)$, where $m \geq 4$ and $n \geq 3$

let for $\alpha \in [0,1]$

Case I:

If for any $\alpha \in [0,1]$, $\chi_a^\alpha (\tilde{P}_m) = 3$ and $\chi_a^\alpha (\tilde{S}_n) = n-2$

Then $\chi_a^\alpha (\tilde{S}_n) = n-2$, where $n \geq 3$

Since $\widetilde{P}_m^\alpha \Pi \widetilde{S}_n^\alpha = 2\,(n-2)$.

Then $\chi_a^\alpha (\tilde{P}_m \Pi \tilde{S}_n) = 2\,(n-2)$, where $m \geq 4$ and $n \geq 3$.

Case II:

If for any $\alpha \in [0,1]$, $\chi_a^\alpha (\tilde{P}_m) = 1$ and $\chi_a^\alpha (\tilde{S}_n) = n-1$

Then $\chi_a^\alpha (\tilde{P}_m) = 1$, where $m \geq 4$

Since $\widetilde{P}_m^\alpha \Pi \widetilde{S}_n^\alpha = n-1$ and are not adjacent to each other.

Then $\chi_a^\alpha (\tilde{P}_m \Pi \tilde{S}_n) = n-1$, where $m \geq 4$ and $n \geq 3$.

Case III:

If for any $\alpha \in [0,1]$ $\chi_a^\alpha (\tilde{P}_m) = 2$ and $\chi_a^\alpha (\tilde{S}_n) = n-1$

If $\widetilde{P}_m^\alpha = 2$ and not adjacent to any other vertex for m ≥ .

Since $\chi_a^\alpha\ (\tilde{\mathrm{P}}_m) = 2$, where $m \geq 4$

Then $\chi_a^\alpha\ (\tilde{\mathrm{P}}_m \ \Pi\ \tilde{\mathrm{S}}_n) = 2\ (n-1)$, where $m \geq 4$ and $n \geq 3$

$\widetilde{P}_m^\alpha\ \Pi\ \tilde{S}_n^\alpha$ has at least two vertices which have the maximum degree 2 (n-1) and are not adjacent to each other. Therefore, according to adjacent vertex, distinguishing proper edge coloring and algorithm $\widetilde{P}_m^\alpha\ \Pi\ \tilde{S}_n^\alpha$ needs exactly 2 (n-1) colors.

Case IV:

If for any α ∈ [0,1], $\chi_a^\alpha\ (\tilde{\mathrm{P}}_m) = 2$ and $\chi_a^\alpha\ (\tilde{\mathrm{S}}_2) = n-2$

If for any $m \geq 4$ $\widetilde{P}_m^\alpha$ has at least one vertex which has the maximum degree 2 and not adjacent to any other vertex.

Then $\chi_a^\alpha\ (\tilde{\mathrm{P}}_m) = 2$, where $m \geq 4$

Similarly, for any $n \geq 3$ $\tilde{S}_n^\alpha$ has only one vertex which has the maximum degree n-2 and is adjacent to all other vertices except one.

Then $\chi_a^\alpha\ (\tilde{\mathrm{S}}_n) = n-2$, where $n \geq 3$

Since $\widetilde{P}_m^\alpha\ \Pi\ \tilde{S}_n^\alpha = 2\ (n\text{-}2)$ and are not adjacent to each other.

Then $\chi_a^\alpha\ (\tilde{\mathrm{P}}_m \ \Pi\ \tilde{\mathrm{S}}_n) = 2\ (n-2)$, where $m \geq 4$ and $n \geq 3$.

Case V:

If for any α ∈ [0,1], $\chi_a^\alpha\ (\tilde{\mathrm{P}}_m) = 1$ and $\chi_a^\alpha\ (\tilde{\mathrm{S}}_n) = n-k$

Then for any $m \geq 4$ $\widetilde{P}_m^\alpha$ has at least one vertex which has the maximum degree 1 and is not adjacent to any other vertex.

Then $\chi_a^\alpha\ (\tilde{\mathrm{P}}_m) = 1$, where $m \geq 4$

Similarly, for any $n \geq 3$ $\tilde{S}_n^\alpha$ has only one vertex which has the maximum degree n-k for some $k \geq 0$ and is adjacent to all other vertices. Therefore, according to adjacent vertex, distinguishing proper edge coloring $\tilde{S}_n^\alpha$ needs exactly n-k colors.

So $\chi_a^\alpha\ (\tilde{\mathrm{S}}_n) = n-k$, where $n \geq 3$ and $k \geq 0$

Since $\widetilde{P}_m^\alpha\ \Pi\ \tilde{S}_n^\alpha$ has at least two vertices which have the maximum degree n-k and are not adjacent to each other.

Then $\chi_a^\alpha\ (\tilde{\mathrm{P}}_m \ \Pi\ \tilde{\mathrm{S}}_n) = n-k$, where $m \geq 4$, $n \geq 3$ and $k \geq 0$

Since the graph of the α-cut contains a finite number of edges and only one edge is colored in each iteration of the algorithm, the algorithm will terminate in a finite number of iterations.

Note: In the crisp direct product of a path and a star graph, there is a fixed degree sequence for the vertex distinguishing proper edge chromatic number for adjacent vertices. By contrast, in the fuzzy direct product, there are several possible α-level representations depending on the membership intensity. Thus, this proposed work is based on uncertainties in edge participation, rather than any deterministic structure.

5. Computational complexity

Let m be the number of edges in the α-cut graph. For each edge, the algorithm checks whether it is feasible concerning the previously colored edges connected to it, which takes linear time as well. As a result, the time complexity is $O(m^2)$, which is a polynomial time complexity that can handle graphs of moderate size.

6. Real life applications

Parallel computing: Task parallelism

An influential procedure known as parallel computing divides the workload into smaller tasks that can be accomplished simultaneously by multiple processors functioning together to tackle an issue further rapidly. This technique extraordinarily speeds up calculation, which makes it vital in domains such as scientific investigation, data analysis, and high performance computing, which require to handle huge datasets, compound imitations, or actual processing. By representing tasks and processors as graphs, with vertices addressing tasks or processors within a computing system, edges addressing dependencies or communication networks regarding tasks, and fuzzy membership values signifying vagueness in task implementation periods or communication delays, fuzzy graph theory provides a mathematical basis for modelling and addressing these sorts of parallel computing issues. One particularly useful concept in this context is AVDPEC. The AVDPEC technique is especially effective when applied to the direct product of two specific types of graph structures, fuzzy path and fuzzy star graph structures.

Example 9: Consider a fuzzy path graph $\tilde{\mathrm{P}}_3$, which represents three tasks, T_1, T_2, and T_3, that must be executed in a sequence. The dependencies between these tasks are T_1T_2 and T_2T_3, where T_2 depends on T_1 and T_3 depends on T_2. There is 80% certainty (membership value 0.8) that task T_2 will depend on T_1, and 75% certainty (membership value 0.75) that task T_3 will be dependent on T_2. An AVDPEC in $\tilde{\mathrm{P}}_3$ represents the time slots or processing assignments that ensure tasks T_1, T_2 and T_2, T_3 are scheduled on different processors or at different times to avoid conflicts. Also, consider a fuzzy star graph $\tilde{\mathrm{S}}_5$, which represents a central shared resource R_1 (e.g., memory unit or database) that all tasks T_4, T_5, T_6, and T_7 need to access. The connections between the resource and the tasks indicate the availability of the resource. The fuzzy membership values represent uncertainty in task execution or resource availability. For example, $R_1T_4 = 0.85$, $R_1T_5 = 0.9$, $R_1T_6 = 0.8$, $R_1T_7 = 0.7$. An AVDPEC in $\tilde{\mathrm{S}}_5$ represents distinct time slots or processors for tasks requiring access to the central resource R_1. This avoids resource contention and ensures that all tasks can be processed efficiently. The direct product graph $\tilde{\mathrm{P}}_3\ \Pi\ \tilde{\mathrm{S}}_5$ models a more complex computing environment where sequences of tasks are distributed across multiple processors that share central resources. Both assignment circumstances and common assets that might affect implementation are captured in this product graph. For example, arranging tasks T_1 and T_2 on distinct processors prevents them from competing for the same asset at the same time and guarantees that job T_2 does not begin until T_1 is accomplished. Edge fuzziness in the direct product $\tilde{\mathrm{P}}_3\ \Pi\ \tilde{\mathrm{S}}_5$ designates the degree of ambiguity or inconsistency in the time or resource requirement needed to complete the jobs linked to these edges. While managing both consecutive circumstances and shared resources, an AVDPEC in $\tilde{P}_3 \Pi \tilde{S}_5$ ensures that tasks are distributed among multiple processors to prevent conflicts.

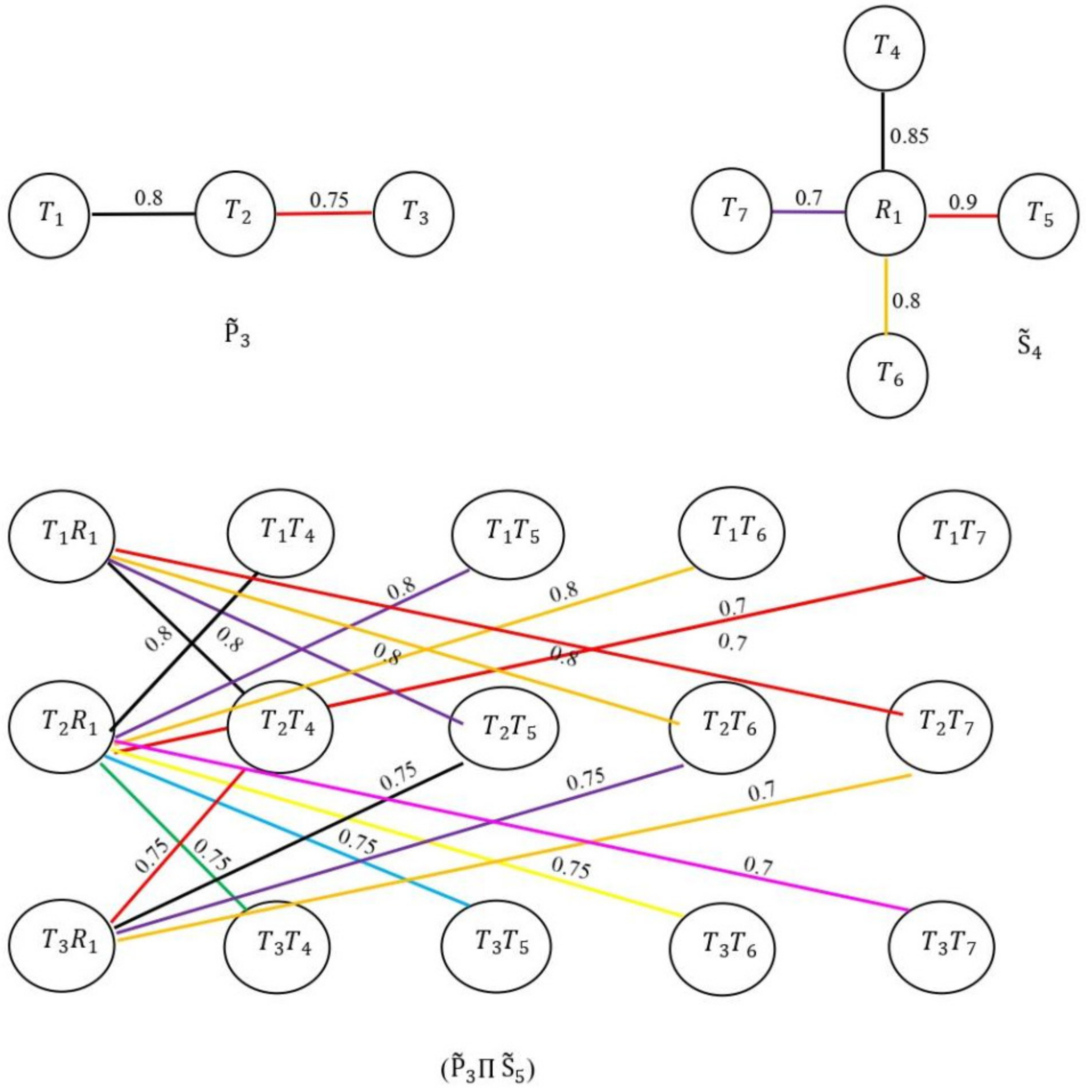


Figure 10. Fuzzy graph of Parallel Computing: Task Parallelism

Load balancing in data networks:

In order to avert overcrowding and ensure efficient data flow, load balancing is vital in data networks such as the internet and data hubs. In data networks such as cloud infrastructure, distributed frameworks, and data centers, load balancing is demonstrated, developed, and implemented extensively utilizing graphs, especially the direct product of fuzzy path and fuzzy star graphs. The associations and communications between several components (such as servers, tasks, data flows, and network nodes) in a data network can be demonstrated numerically employing graphs. The challenge is to assign routes to data packets without overloading any individual path. Data networks consist of multiple routers or servers connected by data paths. Vertices represent routers or servers, edges represent data paths or communication links between routers or servers, and a proper edge coloring algorithm is used to assign different paths to data packets. The key is to guarantee that no two adjacent paths sharing a common router or server are overloaded, thus balancing the load distribution across the network. The capacity and current load of each path are considered as fuzzy membership values. These values, ranging from 0 to 1, indicate the load status of each path: 0 indicates a path with no load, and 1 indicates a path at full capacity. The chromatic number derived from the graph coloring determines the minimum number of routers required to handle the load efficiently.

Example 10: Consider a linear network with four routers: A, B, C, and D. These routers are connected by three paths:

AB, BC, and CD. Assign initial fuzzy membership values to each path based on their capacities and current loads as follows: path AB: 0.3 (low load), path BC: 0.5 (moderate load), and path CD: 0.2 (very low load).

Now assign data packets to different paths according to adjacent vertex distinguishing proper edge coloring. Also consider a star network with four routers: U, V, W, and X. The central router U is connected to all three routers V, W, and X by three different paths: UV, UW, and UX. Assign fuzzy membership values to each path based on their capacities and current loads as follows: path UV: 0.5 (moderate load), path UW: 0.2 (very low load), and path UX: 0.3 (low load).

Use adjacent vertex distinguishing proper edge coloring to assign data packets to different paths so that no two adjacent paths sharing a mutual router are overloaded. As the network expands, combine different network topologies to maintain balanced load distribution. The direct product of $\tilde{P}_4$ and $\tilde{S}_4$ forms a complex network of 16 routers (AU), (AV), (AW), (AX), (BU), (BV), (BW), (BX), (CU), (CV), (CW), (CX), (DU), (DV), (DW) and (DX). These routers are connected by 18 paths (AU) (BV), (AU) (BW), (AU) (BX), (BU) (AV), (BU) (AW), (BU) (AX), (BU) (CV), (BU) (CW), (BU) (CX), (CU) (BV), (CU) (BW), (CU) (BX), (CU) (DV), (CU) (DW), (CU) (DX), (DU) (CV), (DU) (CW) and (DU) (CX). Now assign membership values to different paths of the data network $\tilde{P}_4\ \Pi\ \tilde{S}_4$ according to minimum capacity and loads as follows: path (AU) (BV): 0.3 (low load), path (AU) (BW): 0.2 (very low load), path (AU) (BX): 0.3 (low load), path (BU) (AV): 0.3 (low load), path (BU) (AW): 0.2 (very low load), path (BU) (AX): 0.3 (low load), path (BU) (CV): 0.5 (moderate load), path (BU) (CW): 0.2 (very low load), path (BU) (CX): 0.3 (low load), path (CU) (BV): 0.5 (moderate load), path (CU) (BW): 0.2 (very low load), path (CU) (BX): 0.3 (low load), path (CU) (DV): 0.2 (very low load), path (CU) (DW):0.2 (very low load), path (CU) (DX):0.2 (very low load), path (DU) (CV): 0.2 (very low load), path (DU) (CW): 0.2 (very low load), and path (DU) (CX): 0.2 (very low load).

Assign data packets to different paths according to AVDPEC, using the following algorithm: first, assign data packets based on the minimum membership value (most reliable or least loaded); then balance the load by considering the next minimum membership value incident to the routers of the data network $\tilde{P}_4$ or $\tilde{S}_4$, reuse paths that have already been assigned data packets to ensure efficient utilization, and finally, introduce new paths if necessary to handle additional load.

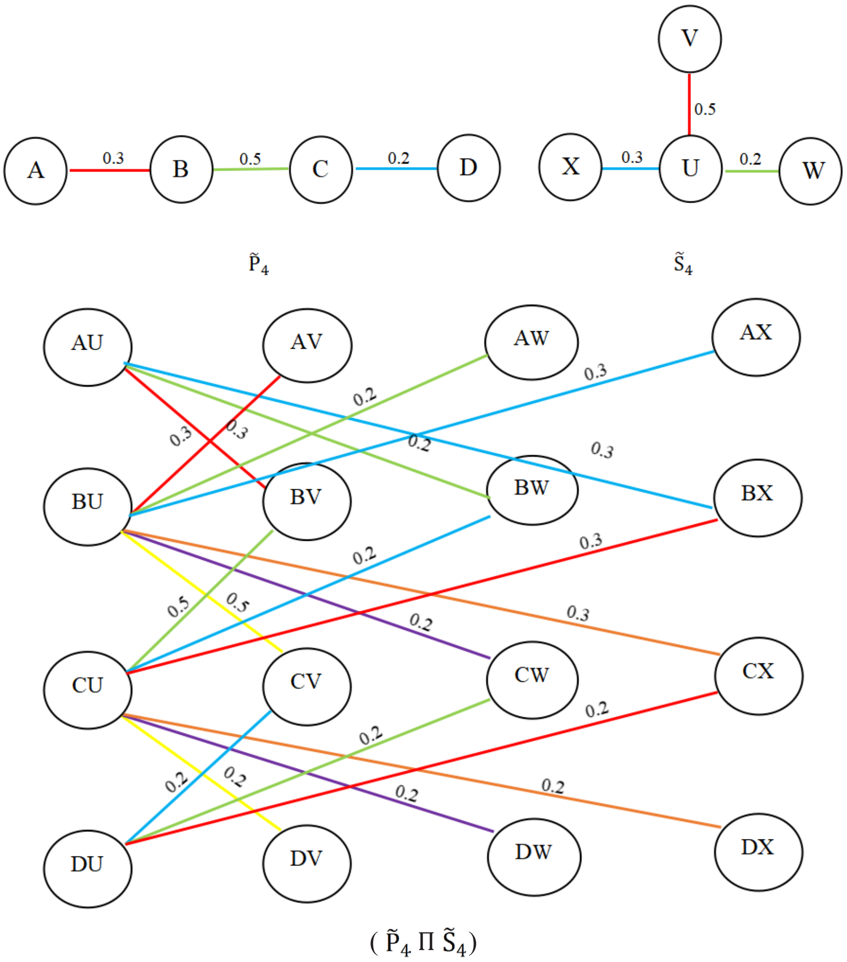


Figure 11. Fuzzy graph for load balancing in data network

7. Conclusion

In this research, a systematic presentation of the AVDPEC of fuzzy graph products is provided, with particular emphasis on fuzzy path and star graphs. This approach establishes a schematic method for determining the AVDPEC of fuzzy direct products, which can be readily generalized to a wide range of intricate fuzzy graph constructions. Our findings demonstrate that this algorithm efficiently calculates the minimal number of colors required to color the direct product so that every pair of neighboring vertices in it can be distinguished while taking into consideration their respective edge memberships. These can be seen to have analytical proofs, involving algorithms and examples that determine its correctness and computational efficiency. The relevance and significance of the study are evident in its applications in the context of giant graphs in which the connectivity may be uncertain, specifically in neural networks, social, and broadcast graphs. The incorporation of fuzziness in membership variety has efficiently enhanced the relevance and applicability of AVDPEC. In this scenario, the "greedy algorithm for the α-cut" in the context of the problem will ensure the appropriateness and vertex distinguishing property of the coloring technique, aiming to use the fewest number of colors, thus directly addressing issues of resource allocation, equality in data graphs, and task allocation in parallel computing networks. Though in the research paper, the methodology is explained with examples in connection with fuzzy direct products of paths and stars, still this proposed framework is quite easy to extend to address complicated graphs. The areas, which can be extended in this research include fuzzy directed graphs, bidirected graphs, hypergraphs, and super hyper graphs. It is important to mention that incorporating these areas will significantly contribute to opening new research directions for specifying uncertainty and multi-attribute relations in complicated graphs, as highlighted in previous literature ([references related to fuzzy directed graphs, hypergraphs, and super hyper graphs]). In summary, the present work has succeeded in moving the frontiers of knowledge regarding the computation of the Average Vertex Degree Product Eigenvalue Coloring Problem in fuzzy graph products. By its success, this research will pave the way not only for practical implementations in network management but also for ways in various researches in fuzzy graph theory.

Authors' contribution

Bacha Khan: Conceptualization, methodology and writing – original draft. Tabasam Rashid: Conceptualization, methodology, validation, supervision and writing – review & editing. Ismat Beg: Conceptualization, methodology, validation and editing.

Conflicts of interest

The authors declare that there is no conflict of interest.

Data availability statement

No data sets were generated or analyzed for the research described in this article.

Funding

No funding used from any source.

References

[1] Zadeh LA. Fuzzy sets. *Information and control.* 1965;8(3): 338-353. doi: 10.1016/S0019-9958(65)90241-X.

[2] Kaufmann A, Swanson DL. Introduction to the theory of fuzzy subsets. *New York: Academic press.* 1975;1: 42-180.

[3] Mutab HMA. Fuzzy Graphs. *Journal of Advances in Mathematics.* 2019;17: 232-247. doi: 10.24297/jam.v17i0.8443.

[4] Rosenfeld A. Fuzzy graphs. In: Zadeh LA, Fu KS, Shimura M. (eds.). *Fuzzy sets and their applications to cognitive and decision processes*. New York: Academic Press; 1975. p. 77-95. doi:10.1016/B978-0-12-775260-0.50008-6.

[5] Mordeson JN, Chang-Shyh P. Operations on fuzzy graphs. *Information Sciences*. 1994;79(3-4): 159-170. doi: 10.1016/0020-0255(94)90116-3.

[6] Molodtsov D. Soft set theory first results. *Computers & Mathematics with Applications*. 1999;37(4-5): 19-31. doi:10.1016/S0898-1221(99)00056-5.

[7] Raut S, Pal M. On chromatic number and perfectness of fuzzy graph. *Information Sciences*. 2022;597: 392-411. doi: 10.1016/j.ins.2022.03.050.

[8] Gong Z, Zhang J. Chromatic Number of fuzzy graphs: Operations, fuzzy graph coloring, and applications. *Axioms*. 2022;11(12): 697. doi:10.3390/axioms11120697.

[9] Sebastian A, Mathew S, Mordeson JN. A new fuzzy graph parameter for the comparison of human trafficking chains. *Fuzzy Sets and Systems*. 2022;450: 27-46. doi: 10.1016/j.fss.2022.04.016.

[10] Malaguti E, Monaci M, Toth P. An exact approach for the vertex coloring problem. *Discrete Optimization*. 2011;8(2): 174-190. doi: 10.1016/j.disopt.2010.07.005.

[11] Maheswari AU, Purnalakshimi AS, Samuvel BJ. Rainbow dominator coloring for special graphs. *International Journal of Mechanical Engineering*.

2022;7(5): 125-133.

[12] Deepa P, Srinivasan P, Sundarakannan M. Local edge coloring of graphs. *AKCE International Journal of Graphs and Combinatorics*. 2021;18(1): 29-32. doi: 10.1080/09728600.2021.1915722.

[13] Hadiputra FF, Maryati TK. A note on local edge antimagic chromatic number of graphs. *Proyecciones (Antofagasta).* 2024;43(2): 447-458. doi: 10.22199/issn.0717-6279-6014.

[14] Petruševski M, Škrekovski R. Proper edge-colorings with a rich neighbor requirement. *Discrete Mathematics.* 2024;347(3): 113803. doi: 10.1016/j.disc.2023.113803.

[15] Zhang Z, Liu L, Wang J. Adjacent strong edge coloring of graphs. *Applied Mathematics Letters*. 2002;15(5): 623-626. doi: 10.1016/S0893-9659(02)80015-5.

[16] Zhang ZF, Woodall DR, Yao B, Li JW, Chen XE, Bian L. Adjacent strong edge colorings and total colorings of regular graphs. *Science in China Series A: Mathematics.* 2009;52(5): 973-980. doi: 10.1007/s11425-008-0153-5.

[17] Chen X, Li Z. Adjacent-vertex-distinguishing proper edge colorings of planar bipartite graphs with $\Delta = 9$, 10, or 11. *Information Processing Letters.* 2015;115(2): 263-268. doi: 10.1016/j.ipl.2014.09.025.

[18] Balister PN, Gyori E, Lehel J, Schelp RH. Adjacent vertex distinguishing edge colorings. *Siam Journal on Discrete Mathematics.* 2007;21(1): 237-250. doi: 10.1137/S0895480102414107.

[19] Zhang Z, Chen X, Li J, Yao B, Lu X, Wang J. On adjacent vertex distinguishing total coloring of graphs. *Science in China Series A: Mathematics.* 2005;48(3): 289-299.

[20] Behzad M. Graphs and their chromatic numbers [PhD thesis]. East Lansing, MI: Michigan State University; 1965. doi: 10.25335/j5he-k143.

[21] Fink JF, Straight HJ. A note on path perfect graphs. *Discrete Mathematics.* 1981;33(1): 95-98. doi: 10.1016/0012-365X(81)90262-4.

[22] Qiu K, Akl SG. On some properties of the star graph. *VLSI Design.* 1995;4(2): 389-396. doi: 10.1155/1995/61390.

[23] Nelson AM. Internal direct products and the universal property of direct product groups. *Formalized Mathematics.* 2023;31(1): 101-120. doi: 10.2478/forma-2023-0010.

[24] Gong Z, Zhang C. Adjacent Vertex Distinguishing Coloring of Fuzzy Graphs. *Mathematics.* 2023;11(10): 2233. doi: 10.3390/math11102233.

[25] Fujita T, Smarandache F. Examples of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets in climate change and the proposal of several new concepts. *Climate Change Reports.* 2025;2: 1-18. doi: 10.61356/j.ccr.2025.2485.

[26] Fujita T, Smarandache F. Local-neutrosophic logic and local-neutrosophic sets: incorporating locality with applications. *Multicriteria Algorithms with Applications.* 2025;6: 66-86. doi: 10.61356/j.mawa.2025.6457.

[27] Sudha S , Martin N , Smarandache F. Applications of Extended Plithogenic Sets in Plithogenic Sociogram. *International Journal of Neutrosophic Science*. 2023;20(4): 08-35. doi:10.54216/IJNS.200401.

[28] Zhu S, Liu Z. Distance measures of picture fuzzy sets and interval-valued picture fuzzy sets with their applications. *AIMS Math.* 2023;8(12): 29817-29848. doi: 10.3934/math.20231525.