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Original Research

A novel dominance based pythagorean fuzzy rough set model

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Abstract: The dominance-based rough sets extend conventional rough sets by substituting the equivalence relation with a dominance relation. Nonetheless, the current dominance relations remain overly limited in practical utility, as they consistently require a precise decrease or increase for every attribute to be considered. Indeed, for numerous practical scenarios, it is sufficient to utilize the ascending or descending arrangement of fractional characteristics instead of considering all attributes or focusing only on the overall assessment of objects. This research established two novel dominance relations derived from the observed phenomenon. Subsequently, we formulated a comprehensive rough set models based on these relations, enabling us to define overall assessments and specific criteria for individual attributes. First, we established a broader form of dominance relations and created a rough set model rooted in this generalized dominance concept. We achieved this by employing the Pythagorean fuzzy additive operator to combine the individualized attribute values of every object in Pythagorean fuzzy environments, resulting in an overall evaluation. Next, we introduced a different form of dominance relationship indicated as the "generalized δ-dominance relation," along with the corresponding "generalized δ-dominance rough set model." This is accomplished by integrating a parameter δ, which is within the range of (0, 1), into the general dominance relationship. The inclusion of this parameter allows us to manage the number of attributes that fulfill dominance relationships, leading to the derivation of decision rules encompassing both "at least" and "at most" conditions. The objective is to develop new dominance relations and rough set models in Pythagorean fuzzy settings, including a generalised δ-dominance relation for flexible attribute evaluation. As a result, the models generate effective decision rules under "at least" and "at most" conditions, with numerical examples validating their applicability. The proposed models support decision making in complicated and unpredictable situations by producing exact rules that are helpful in planning, risk analysis, and supply chain management. They can be used by managers to more clearly and transparently assess options, establish priorities, and justify group decisions.

Keywords: Pythagorean Fuzzy Sets, Fuzzy Ordered Information System, Rough Sets, Dominance Relation

1. Introduction

Pawlak [1] introduced the notion of rough sets in 1982 to provide a mathematical framework for addressing imprecision, ambiguity, and uncertainty in data analysis [2-7]. This concept has been widely applied in various domains such as machine learning, pattern identification, decision analysis, processes regulation, knowledge discovery in databases, and expert systems [8-13], yielding successful outcomes in practical problem-solving scenarios.

Currently, investigations into rough sets have brought forth numerous noteworthy topics for consideration. These

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include challenges related to attribute reduction [14-16], models involving approximation operators [17-19], the development of axiomatic systems [20-22], expansions and variations of rough sets [23-26], and more. Within these areas of study, the exploration of rough set generalizataions stands out as a vital and significant aspect of rough set theory. These generalizations have demonstrated their worth as effective tools for tackling a range of practical issues.

Pawlak's rough sets were created to handle datasets in which each object is limited to having a distinct, discrete value for each attribute. Additionally, in these datasets, every attribute can establish either an equivalence relation within the domain of discourse or a partition of discourse. However, this equivalence or partition approach proves to be constraining in many applications because it cannot detect inconsistencies that arise from the assessment of criteria involving attributes with sequential domains, such as product excellence, market position, and debt ratio [24]. Over the last 35 years, various generalized rough set approaches have been developed to address this issue. One particularly significant approach is the dominance–based rough set approach (DRSA) [27-29,38], which replaces the equivalence relation with a dominance relation. This substitution enables the resolution of ranking issues related to criteria. Consequently, DRSA offers a practical solution for addressing the common inconsistencies found in MADM problems involving exemplary decisions.

The DRSA and Pawlak's rough set model [1] have established a strong basis for managing uncertainty and ranking issues in multicriteria decision making (MCDM). Building on this basis, a wide range of studies have expanded fuzzy and rough-set frameworks to include interval-valued, neutrosophic, hypersoft, Q-rung, and Pythagorean fuzzy environments. These studies have also integrated aggregation operators and decision methods such as TOPSIS for real-world selection problems [31]. As an illustration of how distance-based MCDM tools supplement rough-set ranking techniques [32], TOPSIS and its fuzzy variants have been widely used for selection problems in engineering and medicine. Examples of these include medical-clinic selection for diagnosis and the selection of automotive alternatives using intuitionistic fuzzy TOPSIS [33, 34]. In parallel, the mathematical primitives for expressing richer and precise information and for establishing suitable approximations and aggregation procedures are provided by the algebraic and operational foundations of generalized set structures, such as interval-valued neutrosophic hypersoft sets [35, 36], interval-valued fuzzy soft matrices, and neutrosophic hypersoft matrices.

When combining attribute-level information in MCDM and medical diagnosis applications, recent developments in aggregation and interaction operators [30, 37] (such as interaction geometric, Einstein average, and extensions for q-rung orthopair contexts) demonstrate how aggregation behavior can be tuned and how interaction effects can be captured. The usefulness of combining specialized uncertainty models with ranking techniques is demonstrated by application-driven studies, such as the selection of UAVs for precision agriculture using interval-valued q-rung TOPSIS [39-42] and group decision-making under interval-valued probabilistic-linguistic T-spherical fuzzy information for cloud-storage selection [43].

Moreover, research from other practical fields, such as innovative Möbius transformation-based image encryption algorithms, demonstrates the wide range of applications for sophisticated mathematical and fuzzy/neutrosophic tools beyond classical MCDM, and highlights areas where methods can be cross-fertilized [44]. Our approach addresses the need to handle rich interval, orthopair, and neutrosophic information, support both global (overall) and selective (per-attribute) comparisons, and enable flexible aggregation and decision rules for "at least" and "at most" cases in MADM problems. These studies collectively drive our development of generalized dominance and generalized δ-dominance relations within Pythagorean fuzzy ordered information systems.

Current procedures in the domain of DRSA primarily concentrate on extending methods and creating algorithms to reduce attributes [45-47, 30, 48-51]. The concept of DRSA was originally introduced by Greco et al.[27], who established that condition attributes serve as criteria and are prioritized based on preference. Consequently, the acquired knowledge consists of a collection of dominance classes, which represent sets of objects determined by a dominance relationship. As an illustration, Greco et al. [46] introduced the dominance-based rough fuzzy set approach by integrating DRSA within the framework of fuzzy logic. In this particular approach, the approximation of the vague objective is achieved by employing a dominance relationship instead of a fuzzy relationship. In reference [47], the traditional concept of dominance was extended to establish a one-on-one comparison-based dominance relationship. This extension was used to examine the ordinal characteristics of the preferred levels in pairs of objects. Błaszczynski et al. [45, 52] introduced the VC-DRSA, which incorporates the concept of variable precision rough sets from reference [53] into DRSA and it explores various versions of VC-DRSA. These variations yield broader, lower approximations compared to those computed using DRSA. In other words, they define the lower approximation as specific objects that exhibit a robust, though not necessarily causal, relationship with the sets being approximated. This is achieved by adding a parameter that controls the coherence of the objects incorporated within the lower approximations. Hu [54] introduced an algorithm for determining the reductions in the VC-DRSA. This approach applies to interval-valued intuitionistic fuzzy information systems, where objects are characterized by imprecise evaluations. The concept of dominance-based rough approximations and their implementations in this context has been further developed in these references [55, 48-50, 56].

The dominance relationship in the aforementioned study on dominance rough sets are all established based on the notion of the classical dominance relationship. This conventional dominance relation represents a strong preference among elements in a set, often referred to as "outranking," and signifies a preference for one object over another concerning each criterion. However, this conventional dominance relation is somewhat limiting because it requires that objects satisfy ranking criteria for all information system, rather than just partial ones. Consequently, when constructing dominance categories using conventional dominance relations, even if one attribute of an object is less superior compared to another, it will result in the exclusion of the former from the dominance category of the latter. This characteristic can be particularly significant, especially when dealing with MADM [58-60] problems involving large sets of elements.

Novelty and contribution of the study

The study introduces two new dominance relations—generalized dominance and generalized δ-dominance within Pythagorean fuzzy systems, enabling both overall and selective attribute evaluations. It extends DRSA by deriving decision rules for "at least" and "at most" cases, supported by numerical examples, thus broadening rough set applications in MADM.

As widely recognized, in numerous group decision scenarios, our primary focus is often on identifying entities with the most favorable composite evaluations while disregarding their individual attribute values. In other words, even if an entity exhibits subpar values in certain individual attributes, it should still be considered among the optimal choices as long as its composite evaluation surpasses that of its counterparts. Conversely, in many real-world situations, it is imperative not only to identify entities with superior composite evaluations but also to ensure that they outperform others in specific individual attributes. For instance, in the context of a computer audit risk assessment, when seeking entities that outperform audit objects ꭓ in terms of overall evaluation, our task entails calculating the overall evaluations of all entities and selecting those that outperform ꭓ. In specific cases, it is important to identify objects that not only have an overall evaluation that surpasses ꭓ but also outperform ꭓ in individual attributes. To manage the extent to which an object must outperform others across various criteria, we introduced a parameter δ (where $0 < \delta \leq 1$). Taking these two factors into consideration, we established two new dominance relations within Pythagorean fuzzy information systems. These relations account for both the holistic evaluations of objects and the ranking of objects according to specific criteria, rather than evaluating each criterion individually. The Pythagorean operation is employed to combine all attribute values into a single Pythagorean value, which in turn forms the basis for overall evaluations of objects. Initially, this process leads to the creation of a new dominance relationship, known as the generalized dominance relation, by ranking objects based on their overall evaluations. This relationship helps identify the group of objects with superior comprehensive assessments. Next, the generalized δ-dominance relation is established within the framework of dominance Pythagorean fuzzy ordered information systems, and it is determined by incorporating the parameter δ into the generalized dominance relation. Currently, two distinct dominance relations have been developed to align with decision makers' preferences regarding "overall evaluations" and "individual attribute values." This generalized dominance relation is then utilized in the RSA to introduce the concept of a generalized DRSA within the framework of a dominance-based Pythagorean fuzzy ordered information system.

Research gap and motivation of the study

Classical DRSA relies on strict dominance relations, which require all criteria to be satisfied, limiting its usefulness in real-world problems where partial dominance is acceptable. VC-DRSA relaxes this restriction but still bases dominance on conventional definitions, not addressing cases where composite evaluations should outweigh individual weaknesses. There is a lack of models in the existing literature that integrate overall evaluations with selective attribute dominance in fuzzy environments, and insufficient exploration of Pythagorean fuzzy information systems for generalized dominance-based rough sets.

In many decision-making contexts, the focus is on selecting entities with superior composite evaluations, even if they are weaker in some individual attributes. Conversely, in some applications, decision-makers require assurance that entities also outperform in specific individual attributes. To reconcile these perspectives, flexible dominance relations that can model both overall performance and selective attribute strength are needed. This motivated the development of generalized dominance and δ-dominance relations, offering decision-makers more nuanced tools for handling imprecision and uncertainty in MADM problems. Furthermore, we introduced the generalized δ-DRSA by integrating the generalized δ-dominance relationship, which enables us to generate decision rules for both "at least" and "at most" scenarios. We supported these concepts with numerical examples for better understanding. It is important to note that these two novel approaches introduced in this article differ from VC-DRSA. Our models primarily focus on establishing a broader dominance relationship, which can result in more inclusive dominant categories compared to those produced by DRSA. This distinction arises because our novel dominance relations solely emphasize the order of the comprehensive assessment value, irrespective of individual attribute orders, or impose preferences on the order of only certain attributes. Additionally, the $\mathscr{L}$ Approxs in our approach still employ the "inclusion" relation, akin to DRSA. In contrast, VC-DRSA is similar to DRSA in terms of dominance relations, but it introduces the consistency level parameter $1 \in$ (0, 1] to relax the conditions for inclusion in $\mathcal{L}$ Approxs. This parameter governs the consistency of approximations. Consequently, the $\mathcal{L}$ Approxs of VC-DRSA expand upon those of DRSA, as they can encompass objects that, in line with the traditional interpretation, would not be included in the $\mathcal{L}$ Approxs. VC-DRSA can identify reliances that DRSA might overlook and provides a more robust foundation for generating rules.

The organization of this article can be outlined as follows: In Section 2, we presented a concise overview of the initial considerations of our study, which encompass the explanation of Pythagorean fuzzy set notations, fundamental operations associated with Pythagorean fuzzy sets, and a method for determining the dominance classes of objects based on their attribute values. Section 3 is dedicated to the development of a generalized dominance rough set model by defining a broader dominance relation in a pythagorean fuzzy environment. In section 4, we proposed the concept of generalized δ-dominance rough sets by integrating a generalized δ-dominance relation, which allows us to derive "at least" and "at most" decision rules. Finally, we concluded the research in section 5 with a summary of our findings and a glimpse into potential avenues for future research. Figure 1 is added to show the whole pipline of the proposed research work.

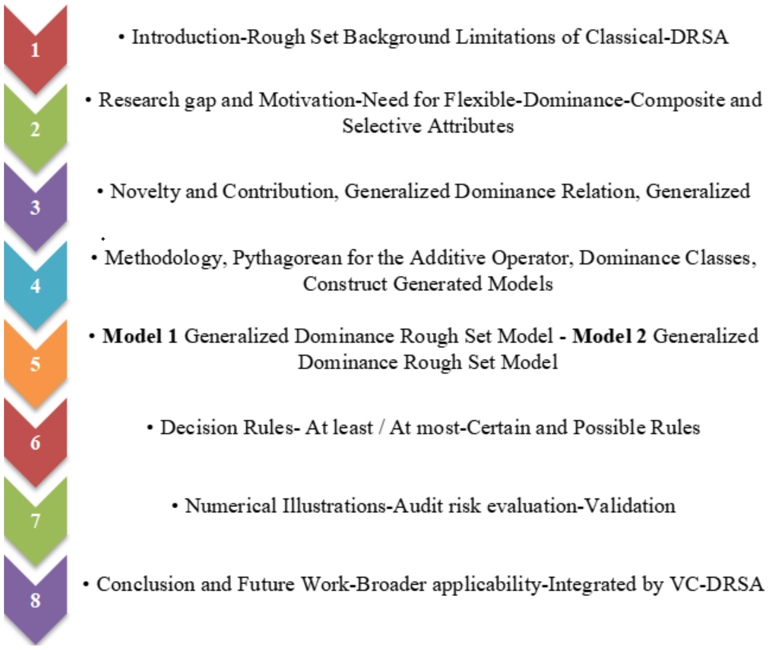

Figure 1. Flowchart of whole research work

2. Preliminaries

This section offers a summary of key information concerning PFSs, IS, PFOIS, and DPFOIS, as well as fundamental concepts relevant to the subject matter of this research. It also touches upon prominent ideas concerning sequential analysis.

A. Pythagorean fuzzy sets

Definition 1: [60] For the universe of discourse U, and for ҳ∈U. The PFS 'P' is defined over U as:

$$P=\left\{\left(ҳ, u_P(ҳ), v_P(ҳ)\right) : u_P, v_P \in [0,1] \; and \; ҳ \in U\right\}$$

where, u_P represents the MG and v_P represents the NMG of ҳ to P, respectively, with $0 \leq \left(u_P(ҳ)\right)^2 + \left(v_P(ҳ)\right)^2 \leq 1$. The indeterminacy degree is

$$\pi_P(ҳ) = \sqrt{1 - \left(u_P(ҳ)\right)^2 - \left(v_P(ҳ)\right)^2}.$$

For convenience, the pythagorean fuzzy number $\left(u_P(ҳ), v_P(ҳ)\right)$ is denoted by (u_P, v_P).

Definition 2: [58-60] Suppose $\beta_i = (u_i, v_i)$, $(1 \leq i \leq 2)$ to be two Pythagorean fuzzy values and $\lambda > 0$, therefore

$$\beta_1 = \beta_2 \Leftrightarrow u_1 = u_2 \wedge v_1 = v_2;$$

$$\beta_1{}^c = (v_1, u_1);$$

$$\lambda\beta_1 = \left(\sqrt{1 - \left(1 - u_1^2\right)^\lambda}, v_1{}^\lambda\right);$$

$$\beta_1{}^\lambda = \left(u_1{}^\lambda, \sqrt{1 - \left(1 - v_1^2\right)^\lambda}\right);$$

$$\beta_1 \cup \beta_2 = \left((max\{u_1, u_2\}, min\{v_1, v_2\}\right);$$

$$\beta_1 \cap \beta_2 = \left((min\{u_1, u_2\}, max\{v_1, v_2\}\right);$$

$$\beta_1 \oplus \beta_2 = \left(\sqrt{u_1^2 + u_2^2 - u_1^2 u_2^2}, v_1 v_2\right);$$

$$\beta_1 \otimes \beta_2 = \left(u_1 u_2, \sqrt{v_1^2 + v_2^2 - v_1^2 v_2^2}\right).$$

Clearly, if β_1 and β_2 are pythagorean fuzzy values, it follows that $\beta_1 \cup \beta_2$ and $\beta_1 \cap \beta_2$ are also PF values. We can readily extend the operations of union, intersection, addition, and multiplication from two Pythagorean fuzzy values to n Pythagorean fuzzy values.

Definition 3: [57] Suppose $\beta_i = (u_i, v_i)$, $(1 \le i \le n)$ to be n Pythagorean fuzzy values, with these values we can establish the union $\cup_{1\le i\le n}\beta_i$, intersection $\cap_{1\le i\le n}\beta_i$, sum $\oplus_{i=1}^{n}\beta_i$ and product $\oplus_{i=1}^{n}\beta_i$ of $\beta_i(1\le i\le n)$ as:

$$\cup_{1\le i\le n}\beta_i = \left(\max_{1\le i\le n} u_i,\ \min_{1\le i\le n} v_i\right);$$

$$\cap_{1\le i\le n}\beta_i = \left(\min_{1\le i\le n} u_i,\ \max_{1\le i\le n} v_i\right);$$

$$\oplus_{i=1}^{n}\beta_i = \left(\sqrt{1 - \prod_{i=1}^{n}\left(1 - u_i^2\right)}, \prod_{i=1}^{n} v_i\right);$$

$$\otimes_{i=1}^{n}\beta_i = \left(\prod_{i=1}^{n} u_i, \sqrt{1 - \prod_{i=1}^{n}\left(1 - v_i^2\right)}\right).$$

Definition 4: Let $\beta_1 = (u_1, v_1)$ and $\beta_2 = (u_2, v_2)$ be two pythagorean fuzzy values, $S(\beta_1) = (u_1)^2 - (v_1)^2$ and $S(\beta_2) = (u_2)^2 - (v_2)^2$ be the score functions of β_1 and β_2, respectively with $S(\beta_1)$ and $S(\beta_2)$ both belongs to [-1, 1]; and $h(\beta_1) = (u_1)^2 + (v_1)^2$ and $h(\beta_2) = (u_2)^2 + (v_2)^2$ be the precisions of β_1 and β_2, respectively, then we can assess the ranking of β_1 and β_2 with these specified criteria.

(1)If $S(\beta_1) < S(\beta_2)$ then $\beta_1 \prec \beta_2$;

(2)If $S(\beta_1) = S(\beta_2)$

a)and $h(\beta_1) = h(\beta_2)$, then $\beta_1 = \beta_2$;

b)and $h(\beta_1) < h(\beta_2)$, then $\beta_1 \prec \beta_2$;

c)and $h(\beta_1) > h(\beta_2)$, then $\beta_1 \succ \beta_2$;

For any two pythagorean fuzzy values, $\beta_1 = (0.6, 0.6)$ and $\beta_2 = (0.4, 0.4)$, $S(\beta_1) = 0.00$, $S(\beta_2) = 0.00$ and $h(\beta_1) = 0.72$, $h(\beta_2) = 0.32$, thus $\beta_1 \succ \beta_2$. So we can develop an approach for prioritizing two items characterized by attribute qualities represented by Pythagorean fuzzy values.

B. Information system

Definition 5: An information system (IS) can be defined as a quadruple $S = (U, AT, V, f)$, where U is a finite non-empty set of objects, AT is a finite nonempty set of attributes, $V = \cup_{q\in AT} V_q$ and V_q is the domain of attribute q, and f: $U \times AT \to V$ is a mapping such that $f(x, q) \in V_q$ for each $q \in AT$, $x \in U$, is an information function.

Definition 6: An IS is known as an OIS if all of its attributes meet the criteria.

C. Pythagorean fuzzy ordered information system

Definition 7: A Pythagorean fuzzy ordered information system (PFOIS) is a quadruple $S = (U, C \cup D, V, f)$, where U is universe of discourse, C is a set of conditional attributes, $D = \{d\}$ is a set of decision attribute, and $C \cap D = \emptyset$. V is the set of all pythagorean fuzzy values, and $V = V_1 \cup V_2$, where V_1 and V_2 are the realms of condition and decision attributes. The information function f is a map from $U \times C \cup D$ onto V, such that $f(x, c) \in V_c$ for all $c \in C$, $V_c \subseteq V_1$, and $f(x, d) \in V_2$ for $D = \{d\}$, where $f(x, c)$ and $f(x, d)$ are pythagorean fuzzy values, denoted by $f(x, c) = (u_c(x), v_c(x))$ and $f(x, d) = (u_d(x), v_d(x))$.

We call $f(x, c)$ the pythagorean fuzzy value of object x, pertaining to the condition attribute c, $f(x, d)$ the pythagorean fuzzy value of x pertaining to the decision attribute d. Specifically, $f(x, c)$ and $f(x, d)$ would degenerate into fuzzy value if $u_c(x) = 1 - v_c(x)$ and $u_d(x) = 1 - v_d(x)$ for every $x \in U$. With this perspective in mind, we view a fuzzy information system as a unique variant of Pythagorean fuzzy information systems. Such information systems sometimes are also called decision-making systems.

When evaluating a decision-making process in a practical context, we consistently examine a binary dominance relationship between items that could potentially excel in certain characteristics within a Pythagorean fuzzy information system. Decision-makers typically consider both increasing and decreasing preferences. An attribute qualifies as a criterion when its domain is ordered in either ascending or descending order of preference. We confine our examination to dominant Pythagorean fuzzy information systems without sacrificing generality.

Evaluating and ranking objects using condition attributes with Pythagorean fuzzy values is a crucial task within dominant Pythagorean fuzzy information systems. To achieve this, we utilize the score function and accuracy function during the ranking procedure for two Pythagorean fuzzy values.

Definition 8: Suppose that PFOIS $=(U, C \cup D, V, f)$, and $B \subseteq C$, for $x, y \in U$ are denoted by $x \preccurlyeq_B y \Leftrightarrow f(x, b) \preccurlyeq f(y, b) \Leftrightarrow f(y, b) \vee f(y, b) = f(y, b)$, $\forall\, b \in B$. Obviously, $\preccurlyeq_B$ is a binary relation in U, that is

$\preccurlyeq_B = \{(x, y) \in U \times U \mid f(x, b) \preccurlyeq f(y, b),\ \forall\, b \in B\}$.

The binary relation described previously is termed a dominance relation in PFOIS.

The dominance class created by the dominance relation, $\preccurlyeq_B$, in terms of $B \subseteq C$, is the set of objects dominating x, *ie* $[x]_B^{\preccurlyeq} = \{y \in U \mid (x, y) \in \preccurlyeq_B\}$ where $[x]_B^{\preccurlyeq}$ defines the group of entities that could potentially dominate x in regards to $B \subseteq C$ and is called the B-dominating set in terms of $x \in U$. In the meantime, the B-dominated set in terms of $x \in U$ can be defined as $[x]_B^{\succcurlyeq} = \{y \in U \mid (y, x) \in \preccurlyeq_B\}$. Similarly, $[x]_B^{\preccurlyeq} = \{y \in U \mid (x, y) \in \preccurlyeq_B\}$ and $[x]_B^{\succcurlyeq} = \{y \in U \mid (y, x) \in \preccurlyeq_B\}$.

D. Dominance-based pythagorean fuzzy ordered information system

This segment explores the set approximation difficulties regarding the dominance-oriented Pythagorean fuzzy relation within the dominance-oriented Pythagorean fuzzy ordered information system (DPFOIS).

Definition 9: Suppose $\hat{S} = (U, C \cup \{d\}, V, f)$ to be DPFOIS, and $B \subseteq C$. The universe U is partitioned into m equivalence classes by d, *i.e.* $U/\{d\} = \{U_1, U_2, \ldots, U_p\}$, where $U_1 \prec U_2 \prec \ldots \prec U_p$ and $U_i \prec U_j$ denotes that $\forall\, x \in U_i$, $y \in U_j \Longrightarrow f(x, d) < f(y, d)$. Denote $U_k^{\preccurlyeq} = \cup_{1 \le j \le k} U_j\ (1 \le k \le p)$, $U_k^{\succcurlyeq} = \cup_{k \le j \le p} U_j (1 \le k \le p)$ and let:

$$\underline{App_B}(U_k^{\preccurlyeq}) = \{x \mid [x]_B^{\succcurlyeq} \subseteq U_k^{\preccurlyeq}\} (1 \le k \le p);$$

$$\overline{App_B}(U_k^{\preccurlyeq}) = \{x \mid [x]_B^{\succcurlyeq} \cap U_k^{\preccurlyeq} \neq \emptyset\}$$

$$= \bigcup_{x \in U_k^{\preccurlyeq}} [x]_B^{\succcurlyeq} (1 \le k \le p);$$

$$BND_B(U_k^{\preccurlyeq}) = \overline{App_B}(U_k^{\preccurlyeq}) - \underline{App_B}(U_k^{\preccurlyeq});$$

$$\underline{App_B}(U_k^{\succcurlyeq}) = \{x \mid [x]_B^{\preccurlyeq} \subseteq U_k^{\succcurlyeq}\} (1 \le k \le p);$$

$$\overline{App_B}(U_k^{\succcurlyeq}) = \{x \mid [x]_B^{\preccurlyeq} \cap U_k^{\succcurlyeq} \neq \emptyset\}$$

$$= \bigcup_{x \in U_k^{\succcurlyeq}} [x]_B^{\preccurlyeq} (1 \le k \le p);$$

$$BND_B(U_k^{\succcurlyeq}) = \overline{App_B}(U_k^{\succcurlyeq}) - \underline{App_B}(U_k^{\succcurlyeq});$$

$$\underline{App}_B^{\preccurlyeq}(x) = \bigcap_{[x]_B^{\preccurlyeq} \cap U_k^{\succcurlyeq} \neq \phi} U_k^{\succcurlyeq};$$

$$\overline{App}_B^{\preccurlyeq}(x) = \bigcap_{[x]_B^{\preccurlyeq} \subseteq U_k^{\succcurlyeq}} U_k^{\succcurlyeq};$$

$$BND_B^{\preccurlyeq}(x) = \overline{App}_B^{\preccurlyeq}(x) - \underline{App}_B^{\preccurlyeq}(x);$$

$$\underline{App}_B^{\succcurlyeq}(x) = \bigcap_{[x]_B^{\succcurlyeq} \cap U_k^{\preccurlyeq} \neq \phi} U_k^{\preccurlyeq}$$

$$\overline{App}_B^{\succcurlyeq}(x) = \bigcap_{[x]_B^{\succcurlyeq} \subseteq U_k^{\preccurlyeq}} U_k^{\preccurlyeq};$$

$$BND_B^{\succcurlyeq}(x) = \overline{App}_B^{\succcurlyeq}(x) - \underline{App}_B^{\succcurlyeq}(x);$$

Then, $\underline{App_B}(U_k^{\preccurlyeq})$, $\overline{App_B}(U_k^{\preccurlyeq})$ and $BND_B(U_k^{\preccurlyeq})$ are called the $\preccurlyeq \mathcal{L}$ Approxs, $\preccurlyeq U$ Approxs and boundary of the dominating class, $U_k^{\preccurlyeq}$ in regards to $\preccurlyeq_B$ with respect to the condition attribute subset B, respectively. $\underline{App_B}(U_k^{\succcurlyeq})$, $\overline{App_B}(U_k^{\succcurlyeq})$ and $BND_B(U_k^{\succcurlyeq})$ are called the $\succcurlyeq \mathcal{L}$ Approxs, $\succcurlyeq U$ Approxs and boundary of the dominated class, $U_k^{\succcurlyeq}$ in regards to $\succcurlyeq_B$ with respect to the condition attribute subset B, respectively.

Whereas, $\underline{App}_B^{\preccurlyeq}(x)$, $\overline{App}_B^{\preccurlyeq}(x)$ and $BND_B^{\preccurlyeq}(x)$ are the $\preccurlyeq \mathcal{L}$ Approxs, $\preccurlyeq U$ Approxs, and boundary of x in regards to $\preccurlyeq_B$ with respect to the condition attribute subset B, respectively. $\underline{App}_B^{\succcurlyeq}(x)$, $\overline{App}_B^{\succcurlyeq}(x)$ and $BND_B^{\succcurlyeq}(x)$ are the $\succcurlyeq \mathcal{L}$ Approxs, $\succcurlyeq U$ Approxs, and boundary of x in regards to $\succcurlyeq_B$ with respect to the condition attribute subset B, respectively.

If $\underline{App_B}(U_k^{\preccurlyeq}) \neq \overline{App_B}(U_k^{\preccurlyeq})$ and $\underline{App_B}(U_k^{\succcurlyeq}) \neq \overline{App_B}(U_k^{\succcurlyeq})$ then $\left(\underline{App_B}(U_k^{\preccurlyeq}), \overline{App_B}(U_k^{\preccurlyeq})\right)$ and $\left(\underline{App_B}(U_k^{\succcurlyeq}), \overline{App_B}(U_k^{\succcurlyeq})\right)$ are referred to as dominating and dominated rough sets.

3. Dominance-based rough set approach for pythagorean fuzzy ordered information system

In this segment, we introduced a generalized rough set approach based on dominance.

The partition of U by decision attribute d is $U/d = \{U_1, U_2, U_3\}$, where $U_1 = \{x_1, x_6, x_8\}$, $U_2 = \{x_2, x_3, x_5, x_9\}$, $U_3 = \{x_4, x_7, x_{10}\}$, then the downward and upward unions are $U_1^{\leqslant} = U_1 = \{x_1, x_6, x_8\}$, $U_2^{\leqslant} = U_1 \cup U_2 = \{x_1, x_2, x_3, x_5, x_6, x_8, x_9\}$, $U_3^{\leqslant} = U_1 \cup U_2 \cup U_3 = U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$). $U_1^{\geqslant} = U_1 \cup U_2 \cup U_3 = U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, $U_2^{\geqslant} = U_2 \cup U_3 = \{x_2, x_3, x_4, x_5, x_7, x_9, x_{10}\}$, $U_3^{\geqslant} = U_3 = \{x_4, x_7, x_{10}\}$. The $\mathcal{L}/\mathrm{U}$ Approxs are in Table 3.

A. Generalized dominance-based rough set model

In this portion, we introduced a comprehensive dominance rough set framework. This framework is established by introducing a novel dominance relation, which relies on the principles of dominance Pythagorean fuzzy decision system.

Suppose $S=(U, C \cup D, V, f)$ to be DPFOIS, the dominace class of $x \in U$ with respect to conditional attributes is described as

$$[x]_B^{\leqslant} = \{y \in U \mid (x, y) \in \leqslant_B\};$$

One can see that it requires $f(x, b) \leqslant f(y, b)$ for each $b \in B \subseteq C$ if $y \in [x]_B^{\leqslant}$. In practical applications, this condition proves to be overly stringent, especially when dealing with large datasets. It can, at times, result in the omission of valuable information in decision-making scenarios. For instance, within the realm of social choice theory [53], the concept of a "dominance class" denoted as $[x]_B^{\leqslant}$ may overlook professional candidates with specialized expertise. Additionally, when considering information granularity, this "dominance class" can yield a more refined level of detail, which in turn escalates computational demands in extensive datasets. Furthermore, the increased granularity can generate a multitude of decision rules, causing complications for decision-makers. This issue will be further elucidated in Example 1.

Table 1. Dominance-based Pythagorean fuzzy ordered information system

U/C	c_1	c_2	c_3	c_4	c_5	*d*
x_1	(0.5, 0.4)	(0.3, 0.4)	(0.8, 0.2)	(0.4, 0.5)	(0.7, 0.2)	1
x_2	(0.3, 0.5)	(0.4, 0.5)	(0.6, 0.3)	(0.4, 0.5)	(0.6, 0.3)	2
x_3	(0.3, 0.5)	(0.4, 0.7)	(0.7, 0.1)	(0.4, 0.5)	(0.7, 0.3)	2
x_4	(0.3, 0.5)	(0.1, 0.8)	(0.4, 0.5)	(0.2, 0.6)	(0.8, 0.1)	3
x_5	(0.7, 0.4)	(0.4, 0.5)	(0.9, 0.1)	(0.4, 0.5)	(0.7, 0.2)	2
x_6	(0.5, 0.7)	(0.6, 0.7)	(0.7, 0.2)	(0.5, 0.4)	(0.6, 0.4)	1
x_7	(0.5, 0.4)	(0.4, 0.5)	(0.7, 0.2)	(0.5, 0.6)	(0.7, 0.1)	3
x_8	(0.4, 0.6)	(0.4, 0.6)	(0.9, 0.1)	(0.7, 0.3)	(0.8, 0.2)	1
x_9	(0.5, 0.7)	(0.7, 0.3)	(0.9, 0.1)	(0.5, 0.6)	(0.9, 0.1)	2
x_{10}	(0.4, 0.7)	(0.3, 0.6)	(0.5, 0.2)	(0.2, 0.9)	(0.4, 0.5)	3

Table 2. Dominating classes $[x]_C^{\leqslant}$ and dominated classes $[x]_C^{\geqslant}$ induced by $\leqslant_C$ and $\geqslant_C$

U	$[x]_C^{\leqslant}$	$[x]_C^{\geqslant}$
x_1	$\{x_1\}$	$\{x_1, x_2, x_3, x_{10}\}$
x_2	$\{x_1, x_2, x_5\}$	$\{x_2, x_{10}\}$
x_3	$\{x_1, x_3, x_5\}$	$\{x_3\}$
x_4	$\{x_4\}$	$\{x_4\}$
x_5	$\{x_5\}$	$\{x_2, x_3, x_5, x_{10}\}$
x_6	$\{x_6\}$	$\{x_6, x_{10}\}$
x_7	$\{x_1, x_7\}$	$\{x_7, x_{10}\}$
x_8	$\{x_8\}$	$\{x_8, x_{10}\}$
x_9	$\{x_9\}$	$\{x_9, x_{10}\}$
x_{10}	$\{x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}\}$	$\{x_{10}\}$

Table 3. $\preccurlyeq$ and $\succcurlyeq$ $\mathcal{L}/U$ Approxs with respect to C

U	$\underline{App}_C^{\preccurlyeq}(x)$	$\overline{App}_C^{\preccurlyeq}(x)$	$\underline{App}_C^{\succcurlyeq}(x)$	$\overline{App}_C^{\succcurlyeq}(x)$
x_1	$U_1^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_2	$U_2^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_2^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_3	$U_2^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_2^{\preccurlyeq}$	$U_2^{\preccurlyeq}$
x_4	$U_3^{\succcurlyeq}$	$U_3^{\succcurlyeq}$	$U_3^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_5	$U_2^{\succcurlyeq}$	$U_2^{\succcurlyeq}$	$U_2^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_6	$U_1^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_7	$U_3^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_3^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_8	$U_1^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_9	$U_2^{\succcurlyeq}$	$U_2^{\succcurlyeq}$	$U_2^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_{10}	$U_3^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_3^{\preccurlyeq}$	$U_3^{\preccurlyeq}$

Example 1

Let us imagine that we are tackling a computer audit risk evaluation challenge, treating it as a decision-making problem with multiple criteria. In this scenario, our goal is to evaluate the risk associated with the audit by considering five different factors. In this issue, the focus is on the audited entities being represented as $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, decision attribute set is $D = d$, where d = "Acceptable Ultimate Computer Auditing Risk", condition attribute set is $C = \{c_1, c_2, c_3, c_4, c_5\}$ where c_1 = "Better Systems Circumstance", c_2 ="Better Systems Control", c_3 = "Safer Finance Data", c_4 =" Credible Auditing Software", c_5 = "Operation Standardization". These condition attributes provide a clear description of the audit candidates outlined in Table 1. Initially, by Definition 8, all the dominating class $[x]_C^{\preccurlyeq}$ and dominated class $[x]_C^{\succcurlyeq}$ of $x \in U$ in regards to condition attribute set C are given in Table 2. From Table 2, we get $[x_2]_C^{\preccurlyeq} = \{x_1, x_2, x_5\}$, it means except x_2, only x_1 and x_5 dominates x_2 in regards to each attribute c_k (k= 1, 2, 3, 4, 5) of C. Yet, according to Definition 3, for every $x_i \in U$, using $C(x_i) = \oplus_{k=1}^{5} f(x_i, c_k)$ signifies the comprehensive assessment of x_i in regards to all condition attributes $c_k \in C$, k=(1,2,…5), where $f(x_i, c_k) = c_k(x_i) = (u_{ck}(x_i), v_{ck}(x_i))$. According to Table 1, $C(x_i) = \oplus_{k=1}^{5} f(x_i, c_k)$ are calculated as

$$C(x_1) = \oplus_{k=1}^{5} f(x_1, c_k) = \left(\sqrt{1 - \prod_{k=1}^{5}\left(1 - u_{c_k}^2(x_1)\right)}, \prod_{k=1}^{5} v_{c_k}(x_1) \right) =$$

$$\left(\begin{array}{c} \sqrt{(1-(1-0.5^2)(1-0.3^2)(1-0.8^2)(1-0.4^2)(1-0.7^2))}, \\ (0.4)(0.4)(0.2)(0.5)(0.2) \end{array} \right)$$

$$= (0.9459, 0.0032).$$

Similarly, we can find

$$C(\text{ӽ}_2)=\oplus_{k=1}^{5}f(\text{ӽ}_2,c_k)=(0.85849,0.01125);$$
$$C(\text{ӽ}_3)=\oplus_{k=1}^{5}f(\text{ӽ}_3,c_k)=(0.91268,0.00525);$$
$$C(\text{ӽ}_4)=\oplus_{k=1}^{5}f(\text{ӽ}_4,c_k)=(0.85933,0.01200);$$
$$C(\text{ӽ}_5)=\oplus_{k=1}^{5}f(\text{ӽ}_5,c_k)=(0.98241,0.00200);$$
$$C(\text{ӽ}_6)=\oplus_{k=1}^{5}f(\text{ӽ}_6,c_k)=(0.93941,0.01568);$$
$$C(\text{ӽ}_7)=\oplus_{k=1}^{5}f(\text{ӽ}_7,c_k)=(0.93653,0.00240);$$
$$C(\text{ӽ}_8)=\oplus_{k=1}^{5}f(\text{ӽ}_8,c_k)=(0.98761,0.00216);$$
$$C(\text{ӽ}_9)=\oplus_{k=1}^{5}f(\text{ӽ}_9,c_k)=(0.99480,0.00126);$$
$$C(\text{ӽ}_{10})=\oplus_{k=1}^{5}f(\text{ӽ}_{10},c_k)=(0.73327,0.03780).$$

By Definition 4 the score function of $C(\text{ӽ}_i)(i=1,2,\ldots,10$ are obtained as follows:
$S(C(\text{ӽ}_1))=0.89472$, $S(C(\text{ӽ}_2))=0.73688$, $S(C(\text{ӽ}_3))=0.83296$, $S(C(\text{ӽ}_4))=0.73830$, $S(C(\text{ӽ}_5))=0.96513$, $S(C(\text{ӽ}_6))=0.88225$, $S(C(\text{ӽ}_7)=0.87708$, $S(C(\text{ӽ}_8))=0.97537$, $S(C(\text{ӽ}_9))=0.98963$, $S(C(\text{ӽ}_{10}))=0.53626$.

By $S(C(\text{ӽ}_i))(i=1,2,\ldots,10)$ the ranking of objects ӽ_i are as follows:

$$\text{ӽ}_9 \succ \text{ӽ}_8 \succ \text{ӽ}_5 \succ \text{ӽ}_1 \succ \text{ӽ}_6 \succ \text{ӽ}_7 \succ \text{ӽ}_3 \succ \text{ӽ}_4 \succ \text{ӽ}_2 \succ \text{ӽ}_{10} \quad (1)$$

Now, let $[\text{ӽ}]_C^{\preccurlyeq}$ denote the collection of entities that dominates ӽ concerning the holistic assessment of condition attribute set C, and $[\text{ӽ}]_C^{\succcurlyeq}$ represent the grouping of entities dominated by ӽ in terms of the overall evaluation of condition attribute set C.

From Table 2, we get $[\text{ӽ}_2]_C^{\preccurlyeq}=\{\text{ӽ}_1,\text{ӽ}_2,\text{ӽ}_5\}$, $[\text{ӽ}_2]_C^{\succcurlyeq}=\{\text{ӽ}_2,\text{ӽ}_{10}\}$. However, Equation 1 shows that $[\text{ӽ}_2]_C^{\preccurlyeq}=\{\text{ӽ}_2,\text{ӽ}_9,\text{ӽ}_8,\text{ӽ}_5,\text{ӽ}_1,\text{ӽ}_6,\text{ӽ}_7,\text{ӽ}_3,\text{ӽ}_4\}$ and $[\text{ӽ}_2]_C^{\succcurlyeq}=\{\text{ӽ}_2,\text{ӽ}_{10}\}$. Obviously, $\text{ӽ}_9,\text{ӽ}_8,\text{ӽ}_6,\text{ӽ}_7,\text{ӽ}_3,\text{ӽ}_4$ do not belong to $[\text{ӽ}_2]_C^{\preccurlyeq}$. The definition of $[\text{ӽ}]_C^{\preccurlyeq}$ proves limiting in various scenarios, as it leads to the exclusion of an object from $[\text{ӽ}]_C^{\preccurlyeq}$, even if just a single attribute value fails to meet the dominance relation $\preccurlyeq_C$. These characteristics hold a significant relevance, particularly when dealing with extensive domains with numerous attributes.

In response to this occurrence, we aim to introduce a novel approach to defining the dominance relationship.

Definition 10: Let $S=(U,C\cup\{d\},V,f)$ be a DPFOIS with $U=\{\text{ӽ}_1,\ldots,\text{ӽ}_n\}$ and $C=\{c_1,\ldots,c_m\}$. Then generalized dominance relation is defined as follows:

$$\preccurlyeq_C^R=\{(\text{ӽ}_i,y_j)\in U\times U \mid \oplus_{k=1}^{m}f(\text{ӽ}_i,c_k)\preccurlyeq\oplus_{k=1}^{m}f(y_j,c_k)\forall c_k\in C\};$$
$$\succcurlyeq_C^R=\{(\text{ӽ}_i,y_j)\in U\times U \mid \oplus_{k=1}^{m}f(\text{ӽ}_i,c_k)\succcurlyeq\oplus_{k=1}^{m}f(y_j,c_k)\forall c_k\in C\}.$$

For any $\text{ӽ}_i\in U$, $[\text{ӽ}_i]_C^{\preccurlyeq R}$ induced by $\preccurlyeq_C^R$ is the set of objects dominate ӽ_i concerning the comprehensive assessment of the condition attribute set C, that is $[\text{ӽ}_i]_C^{\preccurlyeq R}=\{y_j\in U \mid \oplus_{k=1}^{m}f(\text{ӽ}_i,c_k)\preccurlyeq\oplus_{k=1}^{m}f(y_j,c_k)\forall c_k\in C\}$, and $[\text{ӽ}_i]_C^{\preccurlyeq R}$ is known as the generalized dominating class of ӽ_i. Similarly the generalized dominated class of ӽ_i can be described as

$$[\text{ӽ}_i]_C^{\succcurlyeq R}=\{y_j\in U \mid \oplus_{k=1}^{m}f(\text{ӽ}_i,c_k)\succcurlyeq\oplus_{k=1}^{m}f(y_j,c_k)\forall c_k\in C\}.$$

These generalized dominating and dominated classes induced by $\preccurlyeq_C^R$ and $\succcurlyeq_C^R$ form a covering of U, that is,

$$U=\cup_{i=1}^{n}[\text{ӽ}_i]_C^{\preccurlyeq R}=\cup_{i=1}^{n}[\text{ӽ}_i]_C^{\succcurlyeq R}.$$

Remark 1. For any $\text{ӽ}_i,\text{ӽ}_j\in U$,

if $\oplus_{k=1}^{m}f(\text{ӽ}_i,c_k)\preccurlyeq\oplus_{k=1}^{m}f(x_j,c_k)$,,

then $[\text{ӽ}_j]_C^{\preccurlyeq R}\subseteq[\text{ӽ}_i]_C^{\preccurlyeq R}$;

if $\text{ӽ}_j\in[\text{ӽ}_i]_C^{\preccurlyeq R}$, then $[\text{ӽ}_i]_C^{\preccurlyeq R}=\cup_{\text{ӽ}_j\in[\text{ӽ}_i]_C^{\preccurlyeq R}}[\text{ӽ}_j]_C^{\preccurlyeq R}$.

Example 2

Example 2 follows Example 1 calculateing the generalized classes that dominate and are dominated by $\preccurlyeq_C^R$ and $\succcurlyeq_C^R$ in Table 1. All these classes in Table 4 can be obtained. The generalized $\mathcal{L}/U$ Approxs with respect to conditional attributes C are evaluated in Table 5.

From Table 4, it is easy to get $[\text{ӽ}_j]_C^{\preccurlyeq R}\subseteq[\text{ӽ}_i]_C^{\preccurlyeq R}$, if $\text{ӽ}_i\preccurlyeq_C^R\text{ӽ}_j$. Hence, generating generalized dominating and dominated classes by considering the comprehensive assessment of each item across all attributes would prevent the elimination of numerous outstanding items.

From Table 4, it is easy to get $[\text{ӽ}_j]_C^{\preccurlyeq R}\subseteq[\text{ӽ}_i]_C^{\preccurlyeq R}$, if $\text{ӽ}_i\preccurlyeq_C^R\text{ӽ}_j$. Hence, generating generalized dominating and dominated classes by considering the comprehensive assessment of each item across all attributes would prevent the elimination of numerous outstanding items.

Table 4. Generalized dominating classes $[x]_C^{\preccurlyeq R}$ and generalized dominated classes $[x]_C^{\succcurlyeq R}$ by $\preccurlyeq_C^R$ and $\succcurlyeq_C^R$

U	$[x]_C^{\preccurlyeq R}$	$[x]_C^{\succcurlyeq R}$
x_1	$\{x_1, x_5, x_8, x_9\}$	$\{x_1, x_2, x_3, x_4, x_6, x_7, x_{10}\}$
x_2	$\{x_2, x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$	$\{x_2, x_{10}\}$
x_3	$\{x_3, x_1, x_5, x_6, x_7, x_8, x_9\}$	$\{x_3, x_2, x_4, x_{10}\}$
x_4	$\{x_4, x_1, x_3, x_5, x_6, x_7, x_8, x_9\}$	$\{x_4, x_2, x_{10}\}$
x_5	$\{x_5, x_8, x_9\}$	$\{x_5, x_1, x_2, x_3, x_4, x_6, x_7, x_{10}\}$
x_6	$\{x_6, x_1, x_5, x_8, x_9\}$	$\{x_6, x_2, x_3, x_4, x_7, x_{10}\}$
x_7	$\{x_7, x_1, x_5, x_6, x_8, x_9\}$	$\{x_7, x_2, x_3, x_4, x_{10}\}$
x_8	$\{x_8, x_9\}$	$\{x_8, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_{10}\}$
x_9	$\{x_9\}$	$\{x_9, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\}$
x_{10}	$\{x_{10}, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$	$\{x_{10}\}$

Table 5. Generalized $\preccurlyeq$ and $\succcurlyeq$ $\mathcal{L}/U$ Approxs with respect to C

U	$\underline{App}_C^{\preccurlyeq}(x)$	$\overline{App}_C^{\preccurlyeq}(x)$	$\underline{App}_C^{\succcurlyeq}(x)$	$\overline{App}_C^{\succcurlyeq}(x)$
x_1	$U_2^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_2	$U_3^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_2^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_3	$U_3^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_2^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_4	$U_3^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_2^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_5	$U_2^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_6	$U_2^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_7	$U_3^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_2^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_8	$U_2^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_9	$U_2^{\succcurlyeq}$	$U_2^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_{10}	$U_3^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_3^{\preccurlyeq}$	$U_3^{\preccurlyeq}$

Moreover, Table 2 represents that $[x_2]_C^{\preccurlyeq} = \{x_1, x_2, x_5\}$, but as presented in Table 4, $[x_2]_C^{\preccurlyeq R} = \{x_2, x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$. It demonstrates that $x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9$ are all surpass to x_2 in terms of overall assessment, while only the specific attribute values of $x_3, x_4, x_6, x_7, x_8, x_9$ do not satisfy $f(x, c) \preccurlyeq f(y, c)$ for $c \in C$. For example, for object x_6 and x_7, only $f(x_6, c_4) \succ f(x_7, c_4)$, except c_4= "Credible Auditing Software", $f(x_6, c_k) \preccurlyeq f(x_7, c_k)$ holds for any $c_k \in C(k \neq 4)$.

In summary, for any $x_i \in U$, $[x_i]_C^{\preccurlyeq}$ includes only those objects that completely dominate x_i across all attributes of C. This makes it less suitable for addressing multi-attribute collaborative decision-making challenges involving large datasets. Consequently, across various practical contexts, in which there are no specific attribute requirements for objects, it is more practical to focus on the comprehensive assessment of the objects. Particularly, if m=1, then $[x_i]_C^{\preccurlyeq} = [x_i]_C^{\preccurlyeq R}$. This means that $[x_i]_C^{\preccurlyeq} = [x_i]_C^{\preccurlyeq R}$ consistently applies to individual attributes.

Previous analyses are limited to cases where the relative importance of attributes is not taken into account. However, in most real-world decision-making problems, condition attributes often have varying degrees of significance. Recognizing this practical necessity, the proposed approach introduces attribute weighting and defines a weighting dominance relation, offering a more realistic and effective framework than existing DRSA or VC-DRSA methods.

Adding a comparison table to summarize the key differences, advantages, and computational implications between the proposed model and classical DRSA/ VC-DRSA would further highlight its strengths and contributions.

Definition 11: Suppose $S=(U, C\cup\{d\}, V, f)$ to be a DPFOIS with $U=\{x_1, x_2,\ldots, x_n\}$ and $C=\{c_1, \ldots, c_m\}$. Suppose that $W=\{w_1, w_2,\ldots w_m\}$ denotes weight set assigned to condition attribute set, where $w_i (i=1,2,\ldots,m)$ is the weight of c_i and $\sum_{i=1}^{m} w_i = 1$. Then the generalized weighting dominance relation $\preccurlyeq_C^{WR}$ and $\succcurlyeq_C^{WR}$ in S are described as:

$$\preccurlyeq_C^{WR} = \{(x_i, y_j) \in U \times U \mid \oplus_{k=1}^{m} w_k f(x_i, c_k) \preccurlyeq \oplus_{k=1}^{m} w_k f(y_j, c_k), \forall c_k \in C\};$$

$$\succcurlyeq_C^{WR} = \{(x_i, y_j) \in U \times U \mid \oplus_{k=1}^{m} w_k f(x_i, c_k) \succcurlyeq \oplus_{k=1}^{m} w_k f(y_j, c_k), \forall c_k \in C\}.$$

Now by Definition 3,

$$\oplus_{k=1}^{m} w_k f(x_i, c_k) = \left(\sqrt{\left(1 - \prod_{k=1}^{m}\left(1 - u_{c_k}^2(x_i)\right)\right)^{w_k}}, \left(\prod_{k=1}^{m} v_{c_k}(x_i)\right)^{w_k} \right).$$

Similarly, For any $x_i \in U$, $[x_i]_C^{\preccurlyeq WR}$ generated by $\preccurlyeq_C^{WR}$ is the collection of objects that dominates x_i with respect to the comprehensive assessment of condition attributes set C, that is $[x_i]_C^{\preccurlyeq WR} = \{y_j \in U \mid \oplus_{k=1}^{m} w_k f(x_i, c_k) \preccurlyeq \oplus_{k=1}^{m} w_k f(y_j, c_k) \forall c_k \in C\}$, and $[x_i]_C^{\preccurlyeq WR}$ is known as generalized-weighting-dominating class of x_i. Similarly, the generalized-weighting-dominated class of x_i can be described as

$$[x_i]_C^{\succcurlyeq WR} = \{y_j \in U \mid \oplus_{k=1}^{m} w_k f(x_i, c_k) \succcurlyeq \oplus_{k=1}^{m} w_k f(y_j, c_k) \forall c_k \in C\}.$$

Thus, based on Definition 10 and Definition 11 the subsequent characteristics are defined for $[x_i]_C^{\preccurlyeq R}$, $[x_i]_C^{\succcurlyeq R}$, $[x_i]_C^{\preccurlyeq WR}$ and $[x_i]_C^{\succcurlyeq WR}$. For the sake of simplicity, we solely enumerate the properties of $[x_i]_C^{\preccurlyeq R}$ and $[x_i]_C^{\succcurlyeq R}$, which also hold for $[x_i]_C^{\preccurlyeq WR}$ and $[x_i]_C^{\succcurlyeq WR}$.

Theorem 1. Let $S=(U, C\cup\{d\}, V, f)$ be a DPFOIS with $U=\{x_1, x_2,\ldots, x_n\}$ and $C=\{c_1, c_2, \ldots, c_m\}$. $[x_i]_C^{\preccurlyeq R}$ and $[x_i]_C^{\succcurlyeq R}$ are the generalized dominating and dominated classes generated by $\preccurlyeq_C^R$ and $\succcurlyeq_C^R$, $[x_i]_C^{\preccurlyeq}$ and $[x_i]_C^{\succcurlyeq}$ are the dominating and dominated classes generated by $\preccurlyeq_C$ and $\succcurlyeq_C$, respectively. Then

(1) $\preccurlyeq_C^R$ and $\succcurlyeq_C^R$ are reflexive and transitive;

(2) $x_j \in [x_i]_C^{\preccurlyeq R} \Leftrightarrow [x_j]_C^{\preccurlyeq R} \subseteq [x_i]_C^{\preccurlyeq R} \Leftrightarrow [x_j]_C^{\succcurlyeq R} \supseteq [x_i]_C^{\succcurlyeq R}$;

(3) $x_j \in [x_i]_C^{\succcurlyeq R} \Leftrightarrow [x_j]_C^{\succcurlyeq R} \subseteq [x_i]_C^{\succcurlyeq R} \Leftrightarrow [x_j]_C^{\preccurlyeq R} \supseteq [x_i]_C^{\preccurlyeq R}$;

(4) $[x_i]_C^{\preccurlyeq R} = \cup\left\{[x_j]_C^{\preccurlyeq R} \mid x_j \in [x_i]_C^{\preccurlyeq R}\right\}$;

(5) $[x_i]_C^{\preccurlyeq} \subseteq [x_i]_C^{\preccurlyeq R}$, $[x_i]_C^{\succcurlyeq} \subseteq [x_i]_C^{\succcurlyeq R}$;

(6) $[x_i]_C^{\succcurlyeq R} \cap [x_i]_C^{\preccurlyeq R} = \{x_j \in U\} \mid \oplus_{k=1}^{m} f(x_i, c_k) = \oplus_{k=1}^{m} f(x_j, c_k) \forall c_k \in C$;

(7) $U = \cup_{i=1}^{n} [x_i]_C^{\preccurlyeq R} = \cup_{i=1}^{n} [x_i]_C^{\succcurlyeq R}$;

(8) $[x_i]_C^{\preccurlyeq R} = [x_j]_C^{\preccurlyeq R}$ or $[x_i]_C^{\succcurlyeq R} = [x_j]_C^{\succcurlyeq R}$
$\Leftrightarrow \oplus_{k=1}^{m} f(x_i, c_k) = \oplus_{k=1}^{m} f(x_j, c_k), \forall c_k \in C$.

Proof. We exclusively demonstrate (2) and (5) concerning $\preccurlyeq_C^R$; the remaining ones can be directly derived from **Definition 10**.

(2) "$\Longrightarrow$" By **Definition 10**, if $x_j \in [x_i]_C^{\preccurlyeq R}$ then $\oplus_{k=1}^{m} f(x_i, c_k) \preccurlyeq \oplus_{k=1}^{m} f(x_j, c_k)$. Similarly, for any $x_l \in [x_j]_C^{\preccurlyeq R}$ then $\oplus_{k=1}^{m} f(x_j, c_k) \preccurlyeq \oplus_{k=1}^{m} f(x_l, c_k)$. Thus we have $\oplus_{k=1}^{m} f(x_i, c_k) \preccurlyeq \oplus_{k=1}^{m} f(x_l, c_k)$. Hence $x_l \in [x_i]_C^{\preccurlyeq R}$, that is $[x_j]_C^{\preccurlyeq R} \subseteq [x_i]_C^{\preccurlyeq R}$.

"$\Longleftarrow$"It can be easily proved from **Definition 10**.

(5) for any $c_k \in C$, if $x_j \in [x_i]_C^{\preccurlyeq}$, then $f(x_i, c_k) \preccurlyeq f(x_j, c_k)$, thus $\oplus_{k=1}^{m} f(x_i, c_k) \preccurlyeq \oplus_{k=1}^{m} f(x_j, c_k)$. Hence $x_j \in [x_i]_C^{\preccurlyeq R}$, that is $[x_i]_C^{\preccurlyeq} \subseteq [x_i]_C^{\preccurlyeq R}$. Similarly, it can be proved $[x_i]_C^{\succcurlyeq} \subseteq [x_i]_C^{\succcurlyeq R}$.

We are now presenting the generalized DRSA, which includes the incorporation of the generalized dominance relation as defined in **Definition 10**.

Definition 12: Let $S=(U, C\cup\{d\}, V, f)$ be a DPFOIS with $U=\{x_i\}$ $(i=1, 2, \ldots, n)$ and $C=\{c_1, c_2, \ldots, c_m\}$. $[x_i]_C^{\preccurlyeq R}$ and $[x_i]_C^{\succcurlyeq R}$ are the generalized dominating and dominated classes generated by $\preccurlyeq_C^R$ and $\succcurlyeq_C^R$, respectively. The universe U is partioned into p equivalence classes by d that is, $U/\{d\}=\{U_1, U_2, \ldots, U_p\}$, where $U_1 \prec U_2 \prec \ldots \prec U_p$ and $U_i \prec U_j$ denotes that $\forall\, x\in U_i,\ y\in U_j \Rightarrow f(x, d) \preccurlyeq f(y, d)$. Denote $U_k^{\preccurlyeq} = \bigcup_{1\le j\le k} U_j\ (1\le k\le p)$, $U_k^{\succcurlyeq} = \bigcup_{k\le j\le p} U_j (1\le k\le p)$ and let:

$$\underline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) = \left\{x_i \middle| [x_i]_C^{\preccurlyeq R} \subseteq U_k^{\preccurlyeq}\right\}(1\le k\le p);$$

$$\overline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) = \left\{x_i \middle| [x_i]_C^{\succcurlyeq R} \cap U_k^{\preccurlyeq} \ne \emptyset\right\}$$

$$= \bigcup_{x_i\in U_k^{\preccurlyeq}} [x_i]_C^{\succcurlyeq R} (1\le k\le p);$$

$$\mathcal{BND}_C(U_k^{\preccurlyeq}) = \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) - \underline{\mathcal{APP}}_C(U_k^{\preccurlyeq});$$

$$\underline{\mathcal{APP}}_C(U_k^{\succcurlyeq}) = \left\{x_i \middle| [x_i]_C^{\succcurlyeq R} \subseteq U_k^{\succcurlyeq}\right\}(1\le k\le p);$$

$$\overline{\mathcal{APP}}_C(U_k^{\succcurlyeq}) = \left\{x_i \middle| [x_i]_C^{\preccurlyeq R} \cap U_k^{\succcurlyeq} \ne \emptyset\right\}$$

$$= \bigcup_{x_i\in U_k^{\succcurlyeq}} [x_i]_C^{\preccurlyeq R} (1\le k\le p);$$

$$\mathcal{BND}_C(U_k^{\succcurlyeq}) = \overline{\mathcal{APP}}_C(U_k^{\succcurlyeq}) - \underline{\mathcal{APP}}_C(U_k^{\succcurlyeq});$$

$$\underline{\mathcal{APP}}_C^{\preccurlyeq}(x_i) = \bigcap_{[x_i]_C^{\succcurlyeq R} \cap U_k^{\succcurlyeq} \ne \phi} U_k^{\succcurlyeq};$$

$$\overline{\mathcal{APP}}_C^{\preccurlyeq}(x_i) = \bigcap_{[x_i]_C^{\preccurlyeq R} \subseteq U_k^{\succcurlyeq}} U_k^{\succcurlyeq};$$

$$\mathcal{BND}_C^{\preccurlyeq}(x_i) = \overline{\mathcal{APP}}_C^{\preccurlyeq}(x_i) - \underline{\mathcal{APP}}_C^{\preccurlyeq}(x_i);$$

$$\underline{\mathcal{APP}}_C^{\succcurlyeq}(x_i) = \bigcap_{[x_i]_C^{\succcurlyeq R} \cap U_k^{\preccurlyeq} \ne \phi} U_k^{\preccurlyeq};$$

$$\overline{\mathcal{APP}}_C^{\succcurlyeq}(x_i) = \bigcap_{[x_i]_C^{\succcurlyeq R} \subseteq U_k^{\preccurlyeq}} U_k^{\preccurlyeq};$$

$$\mathcal{BND}_C^{\succcurlyeq}(x_i) = \overline{\mathcal{APP}}_C^{\succcurlyeq}(x_i) - \underline{\mathcal{APP}}_C^{\succcurlyeq}(x_i);$$

Then, $\underline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$, $\overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$ and $\mathcal{BND}_C(U_k^{\preccurlyeq})$ are called the $\preccurlyeq\mathcal{L}$ Approxs, $\preccurlyeq$U Approxs and boundary of the dominating class, $U_k^{\preccurlyeq}$ in terms of $\preccurlyeq_C^R$, respectively. $\underline{\mathcal{APP}}_C(U_k^{\succcurlyeq})$, $\overline{\mathcal{APP}}_C(U_k^{\succcurlyeq})$ and $\mathcal{BND}_C(U_k^{\succcurlyeq})$ are the $\succcurlyeq\mathcal{L}$ Approxs, $\succcurlyeq$U Approxs and boundary of the dominated class, $U_k^{\succcurlyeq}$ in terms of $\succcurlyeq_C^R$, respectively.

Whereas, $\underline{\mathcal{APP}}_C^{\preccurlyeq}(x_i)$, $\overline{\mathcal{APP}}_C^{\preccurlyeq}(x_i)$ and $\mathcal{BND}_C^{\preccurlyeq}(x_i)$ are the $\preccurlyeq\mathcal{L}$ Approxs, $\preccurlyeq$U Approxs, and boundary of x_i in terms of $\preccurlyeq_C^R$, respectively. $\underline{\mathcal{APP}}_C^{\succcurlyeq}(x_i)$, $\overline{\mathcal{APP}}_C^{\succcurlyeq}(x_i)$ and $\mathcal{BND}_C^{\succcurlyeq}(x_i)$ are the $\succcurlyeq\mathcal{L}$ Approxs, $\succcurlyeq$U Approxs, and boundary of x_i in terms of $\succcurlyeq_C^R$, respectively.

If $\underline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) \ne \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$ and

$\underline{\mathcal{APP}}_C(U_k^{\succcurlyeq}) \ne \overline{\mathcal{APP}}_C(U_k^{\succcurlyeq})$

then, $\left(\underline{\mathcal{APP}}_C(U_k^{\preccurlyeq}), \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})\right)$

and $\left(\underline{\mathcal{APP}}_C(U_k^{\succcurlyeq}), \overline{\mathcal{APP}}_C(U_k^{\succcurlyeq})\right)$

are known as the generalized dominating and generalized dominated RSs, which are derived from the utilization of generalized dominating and dominated relations. In contrast, we can derive the generalized-weighting dominance RSs by employing $[x_i]_C^{\preccurlyeq WR}$ and $[x_i]_C^{\succcurlyeq WR}$ in a comparable manner.

Remark 2. If $U_k^{\preccurlyeq} \in P(U)$, then we can define

$$\underline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right) = \min_{x_j\in[x_i]_C^{\succcurlyeq R}} U_k^{\preccurlyeq}(x_j);$$

$$\overline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right) = \max_{x_j\in[x_i]_C^{\preccurlyeq R}} U_k^{\preccurlyeq}(x_j).$$

The operators $\underline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right)$ and $\overline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right)$ are denoted as the $\mathcal{L}$/U generalized dominating Pythagorean fuzzy rough approximation operators of $U_k^{\preccurlyeq}$, respectively. So, this pair $\left(\underline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right), \overline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right)\right)$ is termed as a generalized dominating Pythagorean fuzzy rough set.

Based on **Definition 12**, the subsequent properties are valid.

Theorem 2. Let $S=(U, C\cup\{d\}, V, f)$ be a DPFOIS with $U=\{x_i\}$ $(i=1, 2, \ldots, n)$, $C=\{c_1, c_2, \ldots, c_m\}$. $[x_i]_C^{\preccurlyeq R}$ and $[x_i]_C^{\succcurlyeq R}$ are the generalized dominating and dominated classes generated by $\preccurlyeq_C^R$ and $\succcurlyeq_C^R$, respectively.

For any $U_i^{\preccurlyeq}, U_j^{\preccurlyeq} \in P(U)$, the following properties hold:

(1) $\underline{\mathcal{APP}_C}(U_i^{\preccurlyeq}) \subseteq U_i^{\preccurlyeq}, U_i^{\preccurlyeq} \subseteq \overline{\mathcal{APP}_C}(U_i^{\preccurlyeq})$;

(2) $\sim \underline{\mathcal{APP}_C}(U_i^{\preccurlyeq}) = \overline{\mathcal{APP}_C}(U_i^{\preccurlyeq})$,

$\sim \overline{\mathcal{APP}_C}(U_i^{\preccurlyeq}) = \underline{\mathcal{APP}_C}(U_i^{\preccurlyeq})$;

(3) $\underline{\mathcal{APP}_C}(\emptyset^{\preccurlyeq}) = \emptyset, \overline{\mathcal{APP}_C}(\emptyset^{\preccurlyeq}) = \emptyset$;

(4) $\underline{\mathcal{APP}_C}(U^{\preccurlyeq}) = U, \overline{\mathcal{APP}_C}(U^{\preccurlyeq}) = U$;

(5) $\underline{\mathcal{APP}_C}(U_i^{\preccurlyeq} \cap U_j^{\preccurlyeq}) = \underline{\mathcal{APP}_C}(U_i^{\preccurlyeq}) \cap \underline{\mathcal{APP}_C}(U_j^{\preccurlyeq})$,

$\overline{\mathcal{APP}_C}(U_i^{\preccurlyeq} \cup U_j^{\preccurlyeq}) = \overline{\mathcal{APP}_C}(U_i^{\preccurlyeq}) \cup \overline{\mathcal{APP}_C}(U_j^{\preccurlyeq})$;

(6) $\underline{\mathcal{APP}_C}(U_i^{\preccurlyeq} \cup U_j^{\preccurlyeq}) \supseteq \underline{\mathcal{APP}_C}(U_i^{\preccurlyeq}) \cup \underline{\mathcal{APP}_C}(U_j^{\preccurlyeq})$,

$\overline{\mathcal{APP}_C}(U_i^{\preccurlyeq} \cap U_j^{\preccurlyeq}) \subseteq \overline{\mathcal{APP}_C}(U_i^{\preccurlyeq}) \cap \overline{\mathcal{APP}_C}(U_j^{\preccurlyeq})$;

(7) $U_i^{\preccurlyeq} \subseteq U_j^{\preccurlyeq} \Rightarrow \underline{\mathcal{APP}_C}(U_i^{\preccurlyeq}) \subseteq \underline{\mathcal{APP}_C}(U_j^{\preccurlyeq})$,

$\overline{\mathcal{APP}_C}(U_i^{\preccurlyeq}) \subseteq \overline{\mathcal{APP}_C}(U_j^{\preccurlyeq})$.

Proof. The proofs for (1)-(7) can be readily derived using **Definition 12**.

4. Generalized δ-dominance-based rough set model and decision rules exteraction

In this part, we introduced a generalized δ-dominance relation and constructed a corresponding generalized δ-dominance rough set model by incorporating an additional parameter δ into the generalized dominance relation. Then we extracted decision rules from this generalized δ-dominance rough set model

A.Generalized δ-dominance-based rough set model

In this segment, we demonstrated the concept of a broader δ-dominance relationship and proceeded to suggest the generalized δ-DRSA.

As widely recognized, in numerous practical situations, our objective is not just to identify objects with the highest overall assessment, but also to ensure that these objects outperform others in specific attributes. Consequently, the concept of Definition 12 of generalized dominance rough sets no longer holds relevance in such cases.

In the forthcoming passage, Example 1 is consistently used to depict the phenomenon mentioned above. Table 2 shows $[x_6]_C^{\preccurlyeq} = \{x_6\}$ and Table 4 shows $[x_6]_C^{\preccurlyeq R} = \{x_6, x_1, x_5, x_8, x_9\}$. If our goal is to identify objects that not only dominate x_6 in comprehensive assessment but also in at least three attribute values, we should introduce a parameter δ=3/4=0.75 in generalized dominance relation. It is evident that the items meet these two criteria form a distinct set $\left\{x_j | \oplus_{c_k \in C} f(x_6, c_k) \preccurlyeq \oplus_{c_k \in C} f(x_j, c_k) \wedge \frac{|\{c_k | f(x_6, c_k) \preccurlyeq f(x_j, c_k)\}|}{|C|} \geq \frac{3}{4}\right\}$ $=\{x_6, x_1, x_5, x_8, x_9\}$, and denoted as $[x_6]_C^{\preccurlyeq 0.75R}$. Thus, for any $x_i \in [x_6]_C^{\preccurlyeq 0.75R}$, x_i dominates x_6 regarded to comprehensive assessment and at least about three attribute values.

Clearly, it indicates that $[x_6]_C^{\preccurlyeq} \subseteq [x_6]_C^{\preccurlyeq 0.75R} \subseteq [x_6]_C^{\preccurlyeq R}$. As a result, the inclusion of a parameter makes the classification broader in scope compared to the traditional dominance relation.

Definition 13: Let $S=(U, C\cup\{d\}, V, f)$ be a DPFOIS with $U=\{x_i\}$ (i=1, 2, …, n) and $C=\{c_1, c_2, …, c_m\}$. For $\delta \in (0, 1]$, the generalized δ-dominance relation $\preccurlyeq_C^{\delta R}$ and $\succcurlyeq_C^{\delta R}$ are defined as:

$$\preccurlyeq_C^{\delta R} = \left\{(x_i, x_j) \in U \times U | \oplus_{k=1}^m f(x_i, c_k) \preccurlyeq \oplus_{k=1}^m f(x_j, c_k) \wedge \frac{|\{c_k | f(x_i, c_k) \preccurlyeq f(x_j, c_k)\}|}{|C|} \geq \delta \ \forall c_k \in C\right\};$$

$$\succcurlyeq_C^{\delta R} = \left\{(x_i, x_j) \in U \times U | \oplus_{k=1}^m f(x_i, c_k) \succcurlyeq \oplus_{k=1}^m f(x_j, c_k) \wedge \frac{|\{c_k | f(x_i, c_k) \succcurlyeq f(x_j, c_k)\}|}{|C|} \geq \delta \ \forall c_k \in C\right\}.$$

For any $x_i \in U$, the generalized δ-dominating class of x_i generated by $\preccurlyeq_C^{\delta R}$ is represented as $[x_i]_C^{\preccurlyeq \delta R}$, and defined as $[x_i]_C^{\preccurlyeq \delta R} = \left\{x_j \in U | \oplus_{k=1}^m f(x_i, c_k) \preccurlyeq \oplus_{k=1}^m f(x_j, c_k) \wedge, \frac{|\{c_k | f(x_i, c_k) \preccurlyeq f(x_j, c_k)\}|}{|C|} \geq \delta \ \forall c_k \in C\right\}$ where $[x_i]_C^{\preccurlyeq \delta R}$ comprises items that possess both an overall assessment and, at minimum $[\delta.|C|]$ individual attributes surpass those of x_i, where $[\delta.|C|]$ denotes the smallest integer greater than $\delta.|C|$. In the meantime, the generalized δ-dominated class of x_i can be described as $[x_i]_C^{\succcurlyeq \delta R} = \left\{x_j \in U | \oplus_{k=1}^m f(x_i, c_k) \succcurlyeq \oplus_{k=1}^m f(x_j, c_k) \wedge \frac{|\{c_k | f(x_i, c_k) \succcurlyeq f(x_j, c_k)\}|}{|C|} \geq \delta \ \forall c_k \in C\right\}$. $[x_i]_C^{\preccurlyeq \delta R}$ and $[x_i]_C^{\succcurlyeq \delta R}$ are both referred to as the generalized δ-dominance classes of x_i.

The complete set of generalized δ-dominance classes collectively covers U, *i.e.* $U=\cup_{i=1}^n [x_i]_C^{\preccurlyeq \delta R} = \cup_{i=1}^n [x_i]_C^{\succcurlyeq \delta R}$

..Example 3.

Example 3 follows Example 1. Take δ=0.75, by using

Table 6. Generalized δ-dominating classes $[x]_C^{\preccurlyeq\delta R}$ and generalized δ-dominated classes $[x]_C^{\succcurlyeq\delta R}$ induced by $\preccurlyeq_C^{\delta R}$ and $\succcurlyeq_C^{\delta R}$

U	$\overline{[x]_C^{\preccurlyeq 0.75R}}$	$[x]_C^{\succcurlyeq 0.75R}$
x_1	$\{x_1,x_5\}$	$\{x_1,x_2, x_3, x_4, x_6, x_7, x_{10}\}$
x_2	$\{x_2, x_1,x_3, x_5,x_7\}$	$\{x_2, x_{10}\}$
x_3	$\{x_3,x_1,x_5, x_8\}$	$\{x_3, x_2, x_4,x_{10}\}$
x_4	$\{x_4, x_1,x_3, x_5,x_7,x_9\}$	$\{x_4\}$
x_5	$\{x_5\}$	$\{x_5, x_1,x_2,x_3, x_4, x_6, x_7,x_{10}\}$
x_6	$\{x_6, x_1, x_5, x_8,x_9\}$	$\{x_6, x_{10}\}$
x_7	$\{x_7, x_1, x_5, x_9\}$	$\{x_7,x_2,x_4, x_{10}\}$
x_8	$\{x_8\}$	$\{x_8,x_3, x_6,x_{10}\}$
x_9	$\{x_9\}$	$\{x_9, x_4, x_6,x_7,x_{10}\}$
x_{10}	$\{x_{10}, x_1,x_2,x_3, x_5, x_6,x_7,x_8,x_9\}$	$\{x_{10}\}$

Table 7. Generalized ≼ and ≽ *L*/*U* Approxs with respect to *C*

U	$\underline{App}_C^{\preccurlyeq}(x)$	$\overline{App}_C^{\preccurlyeq}(x)$	$\underline{App}_C^{\succcurlyeq}(x)$	$\overline{App}_C^{\succcurlyeq}(x)$
x_1	$U_2^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_2	$U_3^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_2^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_3	$U_2^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_2^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_4	$U_3^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_3^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_5	$U_2^{\succcurlyeq}$	$U_2^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_6	$U_2^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_7	$U_3^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_2^{\preccurlyeq}$	$U_3^{\preccurlyeq}$
x_8	$U_1^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_9	$U_2^{\succcurlyeq}$	$U_2^{\succcurlyeq}$	$\overline{U_1^{\preccurlyeq}}$	$U_3^{\preccurlyeq}$
x_{10}	$U_3^{\succcurlyeq}$	$U_1^{\succcurlyeq}$	$U_3^{\preccurlyeq}$	$U_3^{\preccurlyeq}$

Example 3.

Example 3 follows Example 1. Take δ=0.75, by using **Definition 13**, the generalized δ-dominating classes $[x_i]_C^{\preccurlyeq\delta R}$ and dominated classes $[x_i]_C^{\succcurlyeq\delta R}$ of x_i are gained in Table 6 as below:

From Table 6, we obtained $[x_2]_C^{\preccurlyeq 0.75R}=\{x_2, x_1,x_3, x_5,x_7\}$, which means that x_1,x_3, x_5,x_7 dominates x_2 in terms of overall assessment values and at least [0.75*5]=3 attributes. Particularly, it can be observed that x_1 dominates x_2 regarding $c1$,c_2, c_3,c_4, c_5, x_3 dominates x_2 regarding c_1 ,c_3,c_4, c_5, x_5 dominates x_2 regarding c_1 ,c_2, c_3,c_4, c_5, x_7 dominates x_2 regarding c_1 ,c_2, c_3, c_5.

Of course, the generalized δ-dominance classes specified in Definition 13 possess the subsequent properties:

Theorem 3. Let S=(U, C∪{d}, V,f) be a DPFOIS with U={x_i} (i=1, 2, …, n) and C=c1 ,c2, …, c_m . For δ ∈(0, 1]. Then

(1) $\preccurlyeq_C^{\delta R}$ and $\succcurlyeq_C^{\delta R}$ are are reflexive and transitive;

(2) $x_j \in [x_i]_C^{\preccurlyeq \delta R} \Leftrightarrow [x_j]_C^{\preccurlyeq \delta R} \subseteq [x_i]_C^{\preccurlyeq \delta R}$,

$x_j \in [x_i]_C^{\succcurlyeq \delta R} \Leftrightarrow [x_j]_C^{\succcurlyeq \delta R} \subseteq [x_i]_C^{\succcurlyeq \delta R}$;

(3) $[x_i]_C^{\preccurlyeq \delta R} = \cup\left\{[x_j]_C^{\preccurlyeq \delta R} \mid x_j \in [x_i]_C^{\preccurlyeq \delta R}\right\}$;

(4) $[x_i]_C^{\preccurlyeq} \subseteq [x_i]_C^{\preccurlyeq \delta R} \subseteq [x_i]_C^{\preccurlyeq R}$, $[x_i]_C^{\succcurlyeq} \subseteq [x_i]_C^{\succcurlyeq \delta R} \subseteq [x_i]_C^{\succcurlyeq R}$;

(5) $\delta \in (0.5, 1] \wedge [x_i]_C^{\succcurlyeq \delta R} \cap [x_i]_C^{\preccurlyeq \delta R}$

$= \{x_j \in U \mid f(x_i, c_k) = f(x_j, c_k), \forall\, c_k \in C\}$;

(6) $U = \cup_{i=1}^{n} [x_i]_C^{\preccurlyeq \delta R} = \cup_{i=1}^{n} [x_i]_C^{\succcurlyeq \delta R}$;

(7) $\delta \in (0.5, 1] \wedge [x_i]_C^{\preccurlyeq \delta R} = [x_j]_C^{\preccurlyeq \delta R}\ or\ \left([x_i]_C^{\succcurlyeq \delta R} = [x_j]_C^{\succcurlyeq \delta R}\right)$

$\Rightarrow f(x_i, c_k) = f(x_j, c_k), \forall\, c_k \in C.$

Proof. We only established (5) and (7); the remaining can be directly derived from **Definition 13**.

(8)For any $c_k \in C$, if $x_j \in [x_i]_C^{\succcurlyeq \delta R} \cap [x_i]_C^{\preccurlyeq \delta R}$ then

$$\oplus_{k=1}^{m} f(x_i, c_k) \preccurlyeq \oplus_{k=1}^{m} f(x_j, c_k) \wedge \frac{|\{c_k \mid f(x_i, c_k) \preccurlyeq f(x_j, c_k)\}|}{|C|} \geq \delta$$

and

$$\oplus_{k=1}^{m} f(x_i, c_k) \succcurlyeq \oplus_{k=1}^{m} f(x_j, c_k) \wedge \frac{|\{c_k \mid f(x_i, c_k) \succcurlyeq f(x_j, c_k)\}|}{|C|} \geq \delta$$

It follows that

$$\oplus_{k=1}^{m} f(x_i, c_k) = \oplus_{k=1}^{m} f(x_j, c_k).$$

By Definition 13 if there exists $c_k \in C$ such that $f(x_i, c_k) \neq f(x_j, c_k)$, then we get

$$\frac{|\{c_k \mid f(x_i, c_k) \preccurlyeq f(x_j, c_k)\}|}{|C|} \geq \delta \Leftrightarrow \frac{|\{c_k \mid f(x_i, c_k) \succcurlyeq f(x_j, c_k)\}|}{|C|} < 1 - \delta,$$

Which contradicts to

$$\frac{|\{c_k \mid f(x_i, c_k) \succcurlyeq f(x_j, c_k)\}|}{|C|} \geq \delta, \delta \in (0.5, 1].$$

Thus if $x_j \in [x_i]_C^{\preccurlyeq \delta R} \cap [x_i]_C^{\succcurlyeq \delta R}$ then for every $c_k \in C$, $f(x_i, c_k) = f(x_j, c_k)$.

(7)If $[x_i]_C^{\preccurlyeq \delta R} = [x_j]_C^{\preccurlyeq \delta R}$, then $x_i \in [x_j]_C^{\preccurlyeq \delta R}$ and $x_j \in [x_i]_C^{\preccurlyeq \delta R}$. Thus, there are

$$\oplus_{k=1}^{m} f(x_j, c_k) \preccurlyeq \oplus_{k=1}^{m} f(x_i, c_k) \wedge \frac{|\{c_k \mid f(x_j, c_k) \preccurlyeq f(x_i, c_k)\}|}{|C|} \geq \delta$$

and

$$\oplus_{k=1}^{m} f(x_i, c_k) \preccurlyeq \oplus_{k=1}^{m} f(x_j, c_k) \wedge \frac{|\{c_k \mid f(x_i, c_k) \preccurlyeq f(x_j, c_k)\}|}{|C|} \geq \delta$$

It follows that $\oplus_{k=1}^{m} f(x_i, c_k) \preccurlyeq \oplus_{k=1}^{m} f(x_j, c_k)$. Since $\delta >$ 0.5, we obtained $f(x_i, c_k) = f(x_j, c_k)\ \forall\, c_k \in C$.

A novel rough set model can be presented below by utilizing the generalized δ-dominance relation:

Definition 14. Suppose $S=(U, C\cup\{d\}, V, f)$ to be a DPFOIS with $U=\{x_i\}$ $(i=1, 2, \ldots, n)$ and $C=\{c_1, c_2, \ldots, c_m\}$. Let $[x_i]_C^{\preccurlyeq \delta R}$ be the generalized δ-dominating classes of x_i generated by $\preccurlyeq_C^{\delta R}$ and $[x_i]_C^{\succcurlyeq \delta R}$ be the generalized δ-dominated classes of x_i generated by $\succcurlyeq_C^{\delta R}$. For any $\delta \in 0, 1]$, we defined:

$$\underline{APP}_C(U_k^{\preccurlyeq}) = \left\{x_i \mid [x_i]_C^{\preccurlyeq \delta R} \subseteq U_k^{\preccurlyeq}\right\} (1 \leq k \leq p);$$

$$\overline{APP}_C(U_k^{\preccurlyeq}) = \left\{x_i \mid [x_i]_C^{\succcurlyeq \delta R} \cap U_k^{\preccurlyeq} \neq \emptyset\right\}$$

$$= \bigcup_{x_i \in C U_k^{\preccurlyeq}} [x_i]_C^{\succcurlyeq \delta R} (1 \leq k \leq p);$$

$$BND_C(U_k^{\preccurlyeq}) = \overline{APP}_C(U_k^{\preccurlyeq}) - \underline{APP}_C(U_k^{\preccurlyeq});$$

$$\underline{APP}_C(U_k^{\succcurlyeq}) = \left\{x_i \mid [x_i]_C^{\succcurlyeq \delta R} \subseteq U_k^{\succcurlyeq}\right\} (1 \leq k \leq p);$$

$$\overline{APP}_C(U_k^{\succcurlyeq}) = \left\{x_i \mid [x_i]_C^{\preccurlyeq \delta R} \cap U_k^{\succcurlyeq} \neq \emptyset\right\};$$

$$= \bigcup_{x_i \in U_k^{\succcurlyeq}} [x_i]_C^{\preccurlyeq \delta R} (1 \leq k \leq p);$$

$$BND_C(U_k^{\succcurlyeq}) = \overline{APP}_C(U_k^{\succcurlyeq}) - \underline{APP}_C(U_k^{\succcurlyeq});$$

$$\underline{APP}_C^{\preccurlyeq \delta R}(x_i) = \bigcap_{[x_i]_C^{\preccurlyeq \delta R} \cap U_k^{\succcurlyeq} \neq \phi} U_k^{\succcurlyeq}$$

$$\overline{APP}_C^{\preccurlyeq \delta R}(x_i) = \bigcap_{[x_i]_C^{\preccurlyeq \delta R} \subseteq U_k^{\succcurlyeq}} U_k^{\succcurlyeq};$$

$$BND_C^{\preccurlyeq \delta R}(x_i) = \overline{APP}_C^{\preccurlyeq \delta R}(x_i) - \underline{APP}_C^{\preccurlyeq \delta R}(x_i);$$

$$\underline{APP}_C^{\succcurlyeq \delta R}(x_i) = \bigcap_{[x_i]_C^{\succcurlyeq \delta R} \cap U_k^{\preccurlyeq} \neq \phi} U_k^{\preccurlyeq};$$

$$\overline{\mathcal{APP}}_C^{\succcurlyeq \delta R}(x_i) = \bigcap_{[x_i]_C^{\succcurlyeq \delta R} \subseteq U_k^{\preccurlyeq}} U_k^{\preccurlyeq};$$

$$\mathcal{BND}_C^{\succcurlyeq \delta R}(x_i) = \overline{\mathcal{APP}}_C^{\succcurlyeq \delta R}(x_i) - \underline{\mathcal{APP}}_C^{\succcurlyeq \delta R}(x_i);$$

Then, $\underline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$, $\overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$ and $\mathcal{BND}_C(U_k^{\preccurlyeq})$ are called the $\preccurlyeq\mathcal{L}$ Approxs, $\preccurlyeq U$ Approxs and boundary of the dominating class, $U_k^{\preccurlyeq}$ in terms of $\preccurlyeq_C^{\delta R}$, respectively. $\underline{\mathcal{APP}}_C(U_k^{\succcurlyeq})$, $\overline{\mathcal{APP}}_C(U_k^{\succcurlyeq})$ and $\mathcal{BND}_C(U_k^{\succcurlyeq})$ are the $\succcurlyeq\mathcal{L}$ Approxs, $\succcurlyeq U$ Approxs and boundary of the dominated class, $U_k^{\succcurlyeq}$ in terms of $\succcurlyeq_C^{\delta R}$, respectively.

Whereas, $\underline{\mathcal{APP}}_C^{\preccurlyeq}(x_i)$, $\overline{\mathcal{APP}}_C^{\preccurlyeq}(x_i)$ and $\mathcal{BND}_C^{\preccurlyeq}(x_i)$ are the $\preccurlyeq\mathcal{L}$ Approxs, $\preccurlyeq U$ Approxs, and boundary of x_i in terms of $\preccurlyeq_C^{\delta R}$, respectively. $\underline{\mathcal{APP}}_C^{\succcurlyeq}(x_i)$, $\overline{\mathcal{APP}}_C^{\succcurlyeq}(x_i)$ and $\mathcal{BND}_C^{\succcurlyeq}(x_i)$ are the $\succcurlyeq\mathcal{L}$ Approxs, $\succcurlyeq U$ Approxs, and boundary of x_i in terms of $\succcurlyeq_C^{\delta R}$, respectively.

If $\underline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) \neq \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$ and $\underline{\mathcal{APP}}_C(U_k^{\succcurlyeq}) \neq \overline{\mathcal{APP}}_C(U_k^{\succcurlyeq})$ then, $\left(\underline{\mathcal{APP}}_C(U_k^{\preccurlyeq}), \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})\right)$ and $\left(\underline{\mathcal{APP}}_C(U_k^{\succcurlyeq}), \overline{\mathcal{APP}}_C(U_k^{\succcurlyeq})\right)$ are referred to as generalized δ-dominating and generalized δ-dominated rough sets, this can be achieved by utilizing generalized dominating and dominated relations.

Conversely, we can derive the generalized-weighting dominance rough sets by employing $[x_i]_C^{\preccurlyeq W\delta R}$ and $[x_i]_C^{\succcurlyeq W\delta R}$ in a manner analogous to the aforementioned approach.

Remark 3. If $U_k^{\preccurlyeq} \in P(U)$, then we can define

$$\underline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right) = \min_{x_j \in [x_i]_C^{\preccurlyeq \delta R}} U_k^{\preccurlyeq}(x_j);$$

$$\overline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right) = \max_{x_j \in [x_i]_C^{\preccurlyeq \delta R}} U_k^{\preccurlyeq}(x_j).$$

The operators $\underline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right)$ and $\overline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right)$ are denoted as the $\mathcal{L}/U$ generalized δ-dominating Pythagorean fuzzy rough approximation operators of $U_L^{\preccurlyeq}$, respectively. So, this pair $\left(\underline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right), \overline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right)\right)$ is termed as a a generalized δ-dominating Pythagorean fuzzy rough set of $U_k^{\preccurlyeq}$ regarding $\preccurlyeq_C^{\delta R}$. $\underline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right)$ is the just the degree to which x_i certainly belongs to $U_k^{\preccurlyeq}$, $\overline{\mathcal{APP}}_C\left(U_k^{\preccurlyeq}(x_i)\right)$ is the degree to which x_i possibily belongs to $U_k^{\preccurlyeq}$.

Particularly, if we take $U_k^{\preccurlyeq} = D_j^{\preccurlyeq} = [x_j]_d^{\preccurlyeq R}$ $(i = 1, 2, \ldots, n)$ in Definition 14, then $\underline{\mathcal{APP}}_C(D_j^{\preccurlyeq})$ is called the lower generalized δ-dominating approximation of $D_j^{\preccurlyeq}$ regarding $\preccurlyeq_C^{\delta R}$, whereas $[x_j]_d^{\preccurlyeq R}$ is the generalized dominating class of x_j generated by decision attribute d. Similarly, $\underline{\mathcal{APP}}_C(D_j^{\succcurlyeq})$ will be the lower generalized δ-dominating approximation of $D_j^{\succcurlyeq}$ regarding $\preccurlyeq_C^{\delta R}$. In a similar way $\underline{\mathcal{APP}}_C(D_j^{\preccurlyeq})$ and $\underline{\mathcal{APP}}_C(D_j^{\succcurlyeq})$ could be obtained through $\succcurlyeq_C^{\delta R}$.

Theorem 4. Let $S=(U, C\cup\{d\}, V, f)$ be a DPFOIS with $U=\{x_i\}(i=1, 2, \ldots, n)$ and $C=\{c_1, c_2, \ldots, c_m\}$. Let $[x_i]_C^{\preccurlyeq \delta R}$ be the generalized δ-dominating class of x_i generated by $\preccurlyeq_C^{\delta R}$. Then for $U_k^{\preccurlyeq} \in P(U)$, the subsequent results are valid:

(1) $\underline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) \subseteq U_k^{\preccurlyeq}, U_k^{\preccurlyeq} \subseteq \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$;

(2) $\sim \underline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) = \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$,

$\sim \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) = \underline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$;

(3) $\underline{\mathcal{APP}}_C(\emptyset) = \emptyset, \overline{\mathcal{APP}}_C(\emptyset) = \emptyset$;

(4) $\underline{\mathcal{APP}}_C(U) = U, \overline{\mathcal{APP}}_C(U) = U$;

(5) $\underline{\mathcal{APP}}_C(U \cap U_l^{\preccurlyeq}) = \underline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) \cap \underline{\mathcal{APP}}_C(U_l^{\preccurlyeq})$,

$\overline{\mathcal{APP}}_C(U_k^{\preccurlyeq} \cup U_l^{\preccurlyeq}) = \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) \cup \overline{\mathcal{APP}}_C(U_l^{\preccurlyeq})$

(6) $\underline{\mathcal{APP}}_C(U_k^{\preccurlyeq} \cup U_l^{\preccurlyeq}) \supseteq \underline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) \cup \underline{\mathcal{APP}}_C(U_l^{\preccurlyeq})$,

$\overline{\mathcal{APP}}_C(U_k^{\preccurlyeq} \cap U_l^{\preccurlyeq}) \subseteq \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) \cap \overline{\mathcal{APP}}_C(U_l^{\preccurlyeq})$;

(7) $U_k^{\preccurlyeq} \subseteq U_l^{\preccurlyeq} \Longrightarrow \underline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) \supseteq \underline{\mathcal{APP}}_C(U_l^{\preccurlyeq})$,

$\overline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) \subseteq \overline{\mathcal{APP}}_C(U_l^{\preccurlyeq})$;

(8) If $\delta 1 \leq \delta 2$ then

$\underline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$ *by* $[x_i]_C^{\preccurlyeq \delta_1 R} \subseteq \underline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$ *by* $[x_i]_C^{\preccurlyeq \delta_2 R}$,

and $\overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$ *by* $[x_i]_C^{\preccurlyeq \delta_2 R} \subseteq \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$ *by* $[x_i]_C^{\preccurlyeq \delta_1 R}$.

Proof 1. For any $x_i \in \underline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$, then by Definition 14 $[x_i]_C^{\preccurlyeq \delta R} \subseteq U_k^{\preccurlyeq}$. As $x_i \in [x_i]_C^{\preccurlyeq \delta R}$, this implies that $x_i \in U_k^{\preccurlyeq}$. Thus $\underline{\mathcal{APP}}_C(U_k^{\preccurlyeq}) \subseteq U_k^{\preccurlyeq}$; similarly, for $x_i \in U_k^{\preccurlyeq}$, since $[x_i]_C^{\preccurlyeq \delta R} \cap U_k^{\preccurlyeq} \neq \emptyset$, so $x_i \in \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$. Therefore $U_k^{\preccurlyeq} \subseteq \overline{\mathcal{APP}}_C(U_k^{\preccurlyeq})$.

Proof for (2)-(7) can easily prove directly by Definition 14.

If $\delta 1 \leq \delta 2$ then for any $x_i \in U$ and by Definition 13 $[x_i]_C^{\preccurlyeq_{\delta_2}R} \subseteq [x_i]_C^{\preccurlyeq_{\delta_1}R}$, so by Definition 14 we have $\underline{APP}_C(U_k^{\preccurlyeq})\ by\ [x_i]_C^{\preccurlyeq_{\delta_1}R} \subseteq \underline{APP}_C(U_k^{\preccurlyeq})\ by\ [x_i]_C^{\preccurlyeq_{\delta_2}R}$ and $\overline{APP}_C(U_k^{\preccurlyeq})\ by\ [x_i]_C^{\preccurlyeq_{\delta_2}R} \subseteq \overline{APP}_C(U_k^{\preccurlyeq})\ by\ [x_i]_C^{\preccurlyeq_{\delta_1}R}$.

B. Decision rules extration

In this segment, we explored the topic of obtaining rules within the context of the generalized δ-dominance rough set model in the realm of dominance Pythagorean fuzzy decision systems.

Suppose that $D_j^{\preccurlyeq} = [x_j]_d^{\preccurlyeq R}$ and $D_j^{\succcurlyeq} = [x_j]_d^{\succcurlyeq R}$ $(j = 1, 2, \dots, n)$ are the generalized δ-dominance class of x_j generated by $\preccurlyeq_d^{\delta R}$ and $\succcurlyeq_d^{\delta R}$. Indeed, the outcome of the generalized δ-dominance rough set methodology is a depiction of the information present in the data management solution, expressed as "if... then..." decision rules. Within a Dominance Pythagorean Fuzzy Decision System (DPFDS), four distinct types of dominance rules can be identified.

(1) Certain "at least" rule with the following syntax:

For all $y \in \underline{APP}_C(D_i^{\preccurlyeq})$, if $\left(\oplus_{k=1}^m f(y, c_k) \preccurlyeq \oplus_{k=1}^m f(x, c_k)\right) \wedge (f(y, c_{l1}) \preccurlyeq f(x, c_{l1}) \wedge f(y, c_{l2}) \preccurlyeq f(x, c_{l2}) \wedge \dots \wedge f(y, c_{ln}) \preccurlyeq f(x, c_{ln}))$, then $x \in D_i^{\preccurlyeq}$. In fact, for any $y \in \underline{APP}_C(D_i^{\preccurlyeq})$, then by Definition 14 this implies that $[y]_C^{\preccurlyeq \delta R} \subseteq D_i^{\preccurlyeq}$. On the other hand, as $\left(\oplus_{k=1}^m f(y, c_k) \preccurlyeq \oplus_{k=1}^m f(x, c_k)\right) \wedge (f(y, c_{l1}) \preccurlyeq f(x, c_{l1}) \wedge f(y, c_{l2}) \preccurlyeq f(x, c_{l2}) \wedge \dots \wedge f(y, c_{ln}) \preccurlyeq f(x, c_{ln}))$, then by Definition 13 $y \preccurlyeq_C^{\delta R} x$, it means that $[x]_C^{\preccurlyeq \delta R} \subseteq [y]_C^{\preccurlyeq \delta R}$. Hence $[x]_C^{\preccurlyeq \delta R} \subseteq D_i^{\preccurlyeq}$, it follows that $x \in D_i^{\preccurlyeq}$.

(2)Possible "at least" rule with the following syntax:

For all $y \in \mathcal{BND}_C(D_i^{\preccurlyeq})$, if $\left(\oplus_{k=1}^m f(y, c_k) \preccurlyeq \oplus_{k=1}^m f(x, c_k)\right) \wedge (f(y, c_{l1}) \preccurlyeq f(x, c_{l1}) \wedge f(y, c_{l2}) \preccurlyeq f(x, c_{l2}) \wedge \dots \wedge f(y, c_{ln}) \preccurlyeq f(x, c_{ln}))$, then x could belong to $D_i^{\preccurlyeq}$. If $y \in \mathcal{BND}_C(D_i^{\preccurlyeq})$, then we have $[y]_C^{\preccurlyeq \delta R} \cap D_i^{\preccurlyeq} \neq \emptyset$ and $[y]_C^{\preccurlyeq \delta R} \not\subseteq D_i^{\preccurlyeq}$. From $\left(\oplus_{k=1}^m f(y, c_k) \preccurlyeq \oplus_{k=1}^m f(x, c_k)\right) \wedge (f(y, c_{l1}) \preccurlyeq f(x, c_{l1}) \wedge f(y, c_{l2}) \preccurlyeq f(x, c_{l2}) \wedge \dots \wedge f(y, c_{ln}) \preccurlyeq f(x, c_{ln}))$, there is $[x]_C^{\preccurlyeq \delta R} \subseteq [y]_C^{\preccurlyeq \delta R}$, it is clear that if $[x]_C^{\preccurlyeq \delta R} \subseteq [y]_C^{\preccurlyeq \delta R} \cap D_i^{\preccurlyeq}$, then $[x]_C^{\preccurlyeq \delta R} \subseteq D_i^{\preccurlyeq}$, otherwise $[x]_C^{\preccurlyeq \delta R} \not\subseteq D_i^{\preccurlyeq}$, which means that $[x]_C^{\preccurlyeq \delta R} \cap D_i^{\preccurlyeq} = \emptyset$ or $[x]_C^{\preccurlyeq \delta R} \cap D_i^{\preccurlyeq} \neq \emptyset \wedge [x]_C^{\preccurlyeq \delta R} \cap D_i^{\preccurlyeq} \neq D_i^{\preccurlyeq}$. Hence x could belong to $D_i^{\preccurlyeq}$.

(3)Certain "at most" rule with the following syntax:

Forall $y \in \underline{APP}_C(D_i^{\succcurlyeq})$, if $\left(\oplus_{k=1}^m f(y, c_k) \succcurlyeq \oplus_{k=1}^m f(x, c_k)\right) \wedge (f(y, c_{l1}) \succcurlyeq f(x, c_{l1}) \wedge f(y, c_{l2}) \succcurlyeq f(x, c_{l2}) \wedge \dots \wedge f(y, c_{ln}) \succcurlyeq f(x, c_{ln}))$, then $x \in D_i^{\succcurlyeq}$.

(4)Possible "at least" rule with the following syntax:

Forall $y \in \mathcal{BND}_C(D_i^{\succcurlyeq})$, if $\left(\oplus_{k=1}^m f(y, c_k) \succcurlyeq \oplus_{k=1}^m f(x, c_k)\right) \wedge (f(y, c_{l1}) \succcurlyeq f(x, c_{l1}) \wedge f(y, c_{l2}) \succcurlyeq f(x, c_{l2}) \wedge \dots \wedge f(y, c_{ln}) \succcurlyeq f(x, c_{ln}))$, then x could belong to $D_i^{\succcurlyeq}$; where $\{l_1, l_2, \dots, l_n\} \subseteq \{1, 2, \dots, n\}$ and $\frac{l_n}{n} \geq \delta, i \in \{1, 2, \dots, n\}$.

(3) and (4) can be elucidated in a comparable manner to (1) and (2), respectively.

Consequently, in a DPFDS, and for $D_j^{\preccurlyeq} = [x_j]_d^{\preccurlyeq R}$ or $D_j^{\succcurlyeq} = [x_j]_d^{\succcurlyeq R}$ $(j = 1, 2, \dots, n)$, the rules established based on a hypothesis that objects pertain to $\underline{APP}_C(D_i^{\preccurlyeq})$ $\left(\underline{APP}_C(D_i^{\succcurlyeq})\right)$ are certain "at least" ("at most") rules; the rules established based on a hypothesis that objects pertain to $\mathcal{BND}_C(D_i^{\preccurlyeq})$ $\left(\mathcal{BND}_C(D_i^{\succcurlyeq})\right)$ are possible "at least" ("at most") rules. Thus, the rules established based on a hypothesis that objects pertain to $\overline{APP}_C(D_i^{\preccurlyeq})$ $\left(\overline{APP}_C(D_i^{\succcurlyeq})\right)$ are "at least" ("at most") rules (including certain and possible) "at least" ("at most") rules).

5. Limitations

Without thorough validation on substantial or diverse real-world datasets, this study is primarily restricted to theoretical development and an instructive example. The usefulness of the proposed parameter δ has not been fully investigated, particularly in terms of its sensitivity and robustness in various circumstances.

6. Conclusions and future work

In this article, we established a generalized dominance relationship by evaluating each object with the condition attribute set, ultimately created a generalized dominance-based rough set model in a Pythagorean fuzzy context. We also explored various advantageous characteristics of this model, demonstrating its effectiveness in addressing numerous real-world issues, particularly those involving multi-criteria group decision-making with extensive data sets. In addition, to accommodate specific needs for distinct attribute values in practical scenarios, we proposed an additional rough set model referred to as the generalized δ-dominance rough set model within the Pythagorean fuzzy framework. This model introduces constraints on selected distinct attributes based on the generalized dominance-based rough set model. The introduction of the parameter δ results in the generation of decision rules. Finally, we provided an illustrative example to clarify the concept.

Practical and managerial implications: The proposed models support decision-making in complex and uncertain environments by generating precise rules that are useful in areas such as supply chain management, risk analysis, and planning. Managers can use these models to evaluate alternatives, set priorities, and justify group decisions with greater clarity and transparency.

This work extends classical and intuitionistic rough set approaches by using the Pythagorean fuzzy framework and introducing the novel δ-dominance model with attribute-specific constraints to enhance decision accuracy.

Future studies should investigate the integration of these two novel models with the VC-DRSA, which presents a significant and intriguing topic for further exploration.

Abbreviations

IS	Information System
PFOIS	Pythagorean Fuzzy Ordered Information System
DPFOIS	Dominance-Based Pythagorean Fuzzy Ordered Information System
RSA	Rough Set Approach
DRSA	Dominance-Based Rough Set Approach
MADM	Multi-Attribute Decision Making
VC-DRSA	Variable Consistency Dominance-Based Rough Set Approach
DPFDS	Dominance-Based Pythagorean Fuzzy Decision System
$\mathcal{L}$/U Approxs	Lower/Upper Approximations

Declaration of interests

The authors state that they have not identified any financial or personal conflicts that might have influenced the research detailed in this publication. Furthermore, they assert that there are no conflicting interests related to the publication of this article.

Authors' contribution

All authors contributed equally to the conception, design, analysis, and preparation of this manuscript. All authors have read and approved the final version of the paper.

Data availability statement

No publicly available datasets were used or generated during this study. All data supporting the findings are included within the article.

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